Differential Geometry and its Applications 35 (2014) 323–333

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Hessian structures on deformed exponential families and their conformal structures

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ARTICLE INFO

Article history:
Received 30 November 2013
Received in revised form 28 May 2014
Available online 27 June 2014
Communicated by A. Čap

MSC:
53A15
53A30
53B20

Keywords:
Information geometry
Statistical manifold
Hessian manifold
Tsallis statistics
Deformed exponential family

ABSTRACT

An exponential family is an important class of statistical models in statistical sciences. In information geometry, it is known that an exponential family naturally has dualistic Hessian structures. A deformed exponential family is a statistical model which is a generalization of exponential families. A deformed exponential family naturally has two kinds of dualistic Hessian structures. In this paper, such Hessian geometries are summarized.

In addition, a deformed exponential family has a generalized conformal structure of statistical manifolds. In the case of q-exponential family, which is a special class of deformed exponential families, the family naturally has two kinds of different Riemannian metrics which are obtained from conformal transformations of Hessian metrics. Then it is showed that a q-exponential family is a Riemannian manifold of constant curvature.

1. Introduction

An exponential family is an important statistical model. For example, the set of all normal distributions, all gamma distributions, etc. are exponential families. Such a statistical model is applied to various fields of statistical sciences (cf. [1, Chapters 4–8]). In information geometry, it is known that an exponential family is a Hessian manifold [17, Chapter 6], which is also called a dually flat space [1, Chapter 3] and a flat statistical manifold [7]. In the Hessian geometry, a Riemannian metric and a pair of dualistic affine connections are derived from some potential function, which is formally similar to Kählerian geometry.

A deformed exponential family is a generalization of exponential families, which was introduced in anomalous statistical physics [14]. (See also [15, Part II], and [20, Chapter 3].) In anomalous statistics, we consider two kinds of expectations of random variables, one is the standard expectation and other one

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http://dx.doi.org/10.1016/j.difgeo.2014.06.003

is the expectation with respect to the escort distribution. From these two expectations, we can naturally define two kinds of dualistic Hessian structures on a deformed exponential family. In addition, from the normalization of escort distribution, we can define a Riemannian metric which is conformally equivalent to the Hessian metric.

In this paper, we summarize geometry of such two kinds of Hessian structures and conformal structures on deformed exponential families. Our Hessian geometries give geometric interpretations of generalization of expectation clearly. (See also [9,10] and [11].)

A q-exponential family is a typical example of deformed exponential families. It is important in anomalous statistics and is applied to the theory of complex systems [20]. In particular, a q-normal distribution is a generalization of normal distribution in our framework. Then we often focus geometry of q-normal distributions in this paper. Even though a deformed exponential family has two different conformal structures, we show that the set of all q-normal distributions is regarded as a space of constant curvature in both cases.

2. Statistical manifolds

In this paper, we assume that all the objects are smooth, and a manifold M is an open domain in $\mathbf{R}^n$.

Let (M, h) be a semi-Riemannian manifold. That is, h is a nondegenerate $(0,2)$-tensor field, which is not necessary to be positive definite. In the case we emphasize that the metric is positive definite, we denote it by g rather than h. Let ∇ be a torsion-free affine connection on M. We say that the triplet (M, ∇, h) is a *statistical manifold* if ∇h is totally symmetric [7]. For a statistical manifold (M, ∇, h), the *dual connection* ∇^* of ∇ with respect to h is defined by

$$Xh(Y,Z) = h(\nabla_X Y, Z) + h\bigl(Y, \nabla^*_X Z\bigr),$$

where X, Y and Z are arbitrary vector fields on M. We can check that $(\nabla^*)^* = \nabla$ and $\nabla^{(0)} = (\nabla + \nabla^*)/2$ is the Levi-Civita connection with respect to h. We define a totally symmetric $(0,3)$-tensor filed C by $C = \nabla h$. The tensor field C is called the *cubic form* for (M, ∇, h).

We say that a statistical manifold (M, ∇, h) is *flat* if ∇ is flat. In this case, the dual connection ∇^* is also flat. Hence the quadruplet (M, h, ∇, ∇^*) is called a *dually flat space* [1, Section 3.3]. For a flat statistical manifold (M, ∇, h), suppose that $\{\theta^i\}$ is a ∇-affine coordinate system on M, that is, the connection coefficients Γ_{ij}^k of ∇ vanish everywhere on M. Then there exists a ∇^*-affine coordinate system $\{\eta_i\}$ such that

$$h\left(\frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \eta_j}\right) = \delta_j^i.$$

We call $\{\eta_i\}$ the *dual coordinate system* of $\{\theta^i\}$ with respect to h. The following proposition, called the Legendre duality, is important in information geometry [1, Chapter 3].

Proposition 2.1. *Let (M, ∇, h) be a flat statistical manifold. Suppose that $\{\theta^i\}$ is a ∇-affine coordinate system, and $\{\eta_i\}$ is the dual coordinate system of $\{\theta^i\}$. Then there exist functions ψ and ϕ on M such that*

$$\frac{\partial \psi}{\partial \theta^i} = \eta_i, \qquad \frac{\partial \phi}{\partial \eta_i} = \theta^i, \qquad \psi(p) + \phi(p) - \sum_{i=1}^n \theta^i(p)\eta_i(p) = 0,$$

$$h_{ij} = \frac{\partial^2 \psi}{\partial \theta^i \partial \theta^j}, \qquad h^{ij} = \frac{\partial^2 \phi}{\partial \eta_i \partial \eta_j},$$

where p is an arbitrary point in M, (h_{ij}) is the component matrix of a semi-Riemannian metric h with respect to $\{\theta^i\}$, and (h^{ij}) is the inverse matrix of (h_{ij}).