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A classification of complete Finsler manifolds through the conformal theory of curves

Behroz Bidabad

Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Tehran 15914, Iran

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ABSTRACT

In the first part of this work we are going to touch the conformal theory of curves on Finsler geometry, emphasizing on the notion of circles preserving transformations, recently studied by the present author and Z. Shen. Next, the conformal changes of metrics which leave invariant geodesic circles known as concircular transformation are characterized by a second order differential equation. As an application a classification of complete Finsler manifolds admitting such transformations is obtained.

0. Introduction

The theory of curves including n-dimensional Frenet–Serre formulas, is a basic tool for many progresses in geometry. The reason is quite simple; this formula is exactly the same in the Euclidean, Riemannian and Finslerian spaces. In Riemannian geometry the conformal theory of curves is investigated by many authors, as Fialkow, Kobayashi, Sasaki, Yano, etc. and many interesting and global results are obtained. See for instance [9,18,19,22]. In the first part of this work, we introduce some essential tools to generalize the Fialkow's elaboration in Riemannian spaces for Finslerian spaces.

Using Frenet curvatures one can define a *geodesic circle* in Finsler geometry as a curve with constant curvature and null torsion. Geodesic circles are a natural generalization of straight lines and circles in Euclidean spaces. A conformal change of metric is said to be *concircular* if it maps geodesic circles into geodesic circles.

E-mail address: bidabad@aut.ac.ir.

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If two Finsler structures on a manifold have the same geodesic circles as the point sets, then they are said to be *concircularly related*. Concircular changes are more general than projective changes—change of metric which leaves invariant geodesics—with many comparable properties. These two notions will eventually turn out to be closely related. Similarities between geometric meaning and well known results on concircular change and projective change, let Riemannian geometers make some conjectures and hence played an important role in development of Riemannian geometry.

Much of the practical importance of two concircularly or projectively related ambient spaces derives from the fact that they produce the same physical events. More intuitively in physics a geodesic represents the equation of a motion, which describes a physical phenomena.

Concircular changes are known in physics as conformal Ricci Collineations and they investigate the Einstein–Maxwell field equations in relation with non-null electromagnetic fields. See, for instance, [7,8,14–16].

So far the only known necessary and sufficient condition, characterizing concircular changes was given in [1,10], by presenting a second order partial differential equation determined by Berwald covariant derivative. Failure of metric compatibility in Berwald covariant derivative provides some obstructions for extending certain results of Riemannian geometry, including theory of curves, into the Finsler case. Therefore, we have made some attempts to use Cartan covariant derivative to characterize concircular transformations.

Recently the present author and Z. Shen have defined the notion of a circle on Finsler geometry and studied circle-preserving change of metrics. It is shown that on a Finsler space, a change of metric which keeps geodesic circles is conformal. More precisely, it is proved that if a transformation keeps geodesic circles then a priori it is conformal, cf. [6]. Therefore, every circle-preserving transformation is concircular. Furthermore, in other joint works a necessary and sufficient condition for a vector field to be concircular is obtained, cf., [5,12].

Here in the present work concircular changes of metric are characterized by a second order differential equation and a classification of those Finsler manifolds admitting such change of metric is obtained. More intuitively the following theorems are proved.

Theorem 1. *Let (M,F) be a Finsler manifold. A necessary and sufficient condition for a conformal change $\bar{g}=e^{2\sigma}g$ to be concircular, is the function σ be a solution of the partial differential equation*

$$\nabla_j\sigma_k-\sigma_j\sigma_k=\Phi g_{jk}, \tag{1}$$

where, $\sigma_k=\partial\sigma/\partial x^k$, ∇_k is the Cartan horizontal derivative and Φ is a certain scalar function.

Finally, using a result of [2] the following two classification theorems are obtained.

Theorem 2. *If a connected complete Finsler manifold admits a non-trivial concircular change, then it is conformal to one of the following spaces.*

(a) *An n-dimensional unit sphere in an Euclidean space.*
(b) *An n-dimensional Euclidean space.*
(c) *A direct product $I\times N$ of an open interval I of the real line and an $(n-1)$-dimensional complete Finsler manifold N.*

Theorem 3. *Let (M,F) be a Finsler n-manifold. Then if (M,F) admits a non-trivial concircular change, then it is projectively flat or a direct product $I\times N$ of an open interval I of the real line and an $(n-1)$-dimensional Finsler manifold N.*

Eq. (1) is equivalent to the second order differential equation studied in [4].