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Finsler 2-manifolds with maximal holonomy group of infinite dimension

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ABSTRACT

In this paper we are investigating the holonomy structure of Finsler 2-manifolds. We show that the topological closure of the holonomy group of a certain class of projectively flat Finsler 2-manifolds of constant curvature is maximal, that is isomorphic to the connected component of the diffeomorphism group of the circle. This class of 2-manifolds contains the standard Funk plane of constant negative curvature and the Bryant–Shen-spheres of constant positive curvature. The result provides the first examples describing completely infinite dimensional Finslerian holonomy structures.

1. Introduction

The notion of the holonomy group of a Riemannian or Finslerian manifold can be introduced in a very natural way: it is the group generated by parallel translations along loops with respect to the associated linear, respectively homogeneous (nonlinear) connection. In contrast to the Finslerian case, the Riemannian holonomy groups have been extensively studied. One of the earliest fundamental results is the theorem of Borel and Lichnerowicz [1] from 1952, claiming that the holonomy group of a simply connected Riemannian

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manifold is a closed Lie subgroup of the orthogonal group $O(n)$. By now, the complete classification of Riemannian holonomy groups is known.

The holonomy properties of Finsler spaces is, however, essentially different from the Riemannian one, and it is far from being well understood. Compared to the Riemannian case, only few results are known. In [13] it was proved that the holonomy group of a Finsler manifold of nonzero constant curvature with dimension greater than 2 is not a compact Lie group. In [15] it has been shown that there exist large families of projectively flat Finsler manifolds of constant curvature such that their holonomy groups are not finite dimensional Lie groups. In [14] we characterized the projective Finsler manifolds of constant curvature having infinite dimensional holonomy group. The proofs in the above mentioned papers give estimates for the dimension of tangent Lie algebras of the holonomy group and therefore they do not give direct information about the infinite dimensional structure of the holonomy group.

Until now, perhaps because of technical difficulties, not a single infinite dimensional Finsler holonomy group has been determined. In this paper we provide the first such a description: we show that the topological closure of the holonomy group of a certain class of simply connected, projectively flat Finsler 2-manifolds of constant curvature is $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$, the connected component of the full diffeomorphism group of the circle. This class of Finsler 2-manifolds contains the positively complete standard Funk plane of constant negative curvature (positively complete standard Funk plane), and the complete irreversible Bryant–Shen-spheres of constant positive curvature [17,3]. We remark that for every simply connected Finsler 2-manifold the topological closure of the holonomy group is a subgroup of $\mathsf{Diff}^\infty_+(\mathbb{S}^1)$. Consequently, in the examples mentioned above, the closed holonomy group is maximal. In the proof we use the constructive method developed in [15] to study the Lie algebras of vector fields on the indicatrix which are tangent to the holonomy group.

2. Preliminaries

Throughout this article, M is a C^∞ smooth manifold, $\mathfrak{X}^\infty(M)$ is the vector space of smooth vector fields on M and $\mathsf{Diff}^\infty(M)$ is the group of all C^∞-diffeomorphism of M. The first and the second tangent bundles of M are denoted by (TM, π, M) and (TTM, τ, TM), respectively.

A *Finsler manifold* is a pair $(M, \mathcal{F})$, where the norm function $\mathcal{F}: TM \to \mathbb{R}_+$ is continuous, smooth on $\hat{T}M := TM \setminus \{0\}$, its restriction $\mathcal{F}_x = \mathcal{F}|_{T_xM}$ is a positively homogeneous function of degree one and the symmetric bilinear form

$$g_{x,y}: (u, v) \mapsto g_{ij}(x, y)u^i v^j = \frac{1}{2}\frac{\partial^2 \mathcal{F}^2_x(y + su + tv)}{\partial s \partial t}\bigg|_{t=s=0}$$

is positive definite at every $y \in \hat{T}_xM$.

Geodesics of $(M, \mathcal{F})$ are determined by a system of 2nd order ordinary differential equation $\ddot{x}^i + 2G^i(x, \dot{x}) = 0$, $i = 1, \dots, n$ in a local coordinate system (x^i, y^i) of TM, where $G^i(x, y)$ are given by

$$G^i(x, y) := \frac{1}{4}g^{il}(x, y)\left(2\frac{\partial g_{jl}}{\partial x^k}(x, y) - \frac{\partial g_{jk}}{\partial x^l}(x, y)\right)y^j y^k. \tag{1}$$

A vector field $X(t) = X^i(t)\frac{\partial}{\partial x^i}$ along a curve $c(t)$ is said to be parallel with respect to the associated *homogeneous (nonlinear) connection* if it satisfies

$$D_{\dot{c}}X(t) := \left(\frac{dX^i(t)}{dt} + G^i_j\big(c(t), X(t)\big)\dot{c}^j(t)\right)\frac{\partial}{\partial x^i} = 0, \tag{2}$$

where $G^i_j = \frac{\partial G^i}{\partial y^j}$.