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Metrics on spaces of immersions where horizontality equals normality ☆

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ABSTRACT

We study metrics on shape space of immersions that have a particularly simple horizontal bundle. More specifically, we consider reparametrization invariant Sobolev metrics G on the space $\operatorname{Imm}(M,N)$ of immersions of a compact manifold M in a Riemannian manifold $(N,\bar{g})$. The tangent space $T_f \operatorname{Imm}(M,N)$ at each immersion f has two natural splittings: one into components that are tangential/normal to the surface f (with respect to $\bar{g}$) and another one into vertical/horizontal components (with respect to the projection onto the shape space $B_i(M,N) = \operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ of unparametrized immersions and with respect to the metric G). The first splitting can be easily calculated numerically, while the second splitting is important because it mirrors the geometry of shape space and geodesics thereon. Motivated by facilitating the numerical calculation of geodesics on shape space, we characterize all metrics G such that the two splittings coincide. In the special case of planar curves, we show that the regularity of curves in the metric completion can be controlled by choosing a strong enough metric within this class. We demonstrate in several examples that our approach allows us to efficiently calculate numerical solutions of the boundary value problem for geodesics on shape space.

1. Introduction

Nowadays, the study of the geometry of the space of all immersions of a certain type—including plane curves, space curves and surfaces—is an active area of research with applications in computational anatomy, shape comparison and image analysis, see e.g. [13,27,23,15,18,29,25,12,4]. This space, denoted by $\operatorname{Imm}(M,N)$, consists of smooth immersions from a compact m-dimensional manifold M into an n-dimensional Riemannian manifold $(N,\bar{g})$. The important special case of planar curves corresponds to

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$M = S^1$ and $N = \mathbb{R}^2$. We will always assume that $n \geq m$, in which case $\operatorname{Imm}(M, N)$ is a smooth Fréchet manifold [17,14]; otherwise it is empty.

The notion of shape space in this article is that of unparametrized immersions. This space can be identified with the quotient space

$$B_i(M, N) = \operatorname{Imm}(M, N)/\operatorname{Diff}(M).$$

Here, $\operatorname{Diff}(M)$ denotes the Lie group of all smooth diffeomorphisms on M, which acts smoothly on $\operatorname{Imm}(M, N)$ via composition from the right:

$$\operatorname{Imm}(M, N) \times \operatorname{Diff}(M) \to \operatorname{Imm}(M, N), \qquad (f, \varphi) \mapsto f \circ \varphi.$$

The quotient space $B_i(M, N)$ is not a manifold, but only an orbifold with isolated singularities (see [10] for more information).

Given a reparametrization invariant metric G on $\operatorname{Imm}(M, N)$, we can (under certain conditions) induce a unique Riemannian metric on the quotient space $B_i(M, N)$ such that the projection

$$\pi : \operatorname{Imm}(M, N) \to B_i(M, N) := \operatorname{Imm}(M, N)/\operatorname{Diff}(M) \tag{1}$$

is a Riemannian submersion. A detailed description of this construction is provided in [5, Section 4]. For many metrics, $T\pi$ induces a splitting of the tangent bundle $T\operatorname{Imm}(M, N)$ into a vertical bundle, which is defined as the kernel of $T\pi$, and a horizontal bundle, defined as the G-orthogonal complement of the vertical bundle:

$$T\operatorname{Imm}(M, N) = \ker T\pi \oplus (\ker T\pi)^{\perp, G} =: \operatorname{Ver} \oplus \operatorname{Hor}. \tag{2}$$

If one can lift any curve in $B_i(M, N)$ to a horizontal curve in $\operatorname{Imm}(M, N)$, then there is a one-to-one correspondence between geodesics on shape space $B_i(M, N)$ and horizontal geodesics on $\operatorname{Imm}(M, N)$ [5, Section 4.8]. This constitutes an effective way to compute geodesics on shape space provided that the horizontal bundle is not too complicated. A particularly favorable case is when the splitting in Eq. (2) coincides with the natural splitting into components that are tangential and normal to the immersed surface with respect to the metric $\bar{g}$:

$$T\operatorname{Imm}(M, N) = \operatorname{Tan} \oplus \operatorname{Nor}. \tag{3}$$

If $\pi_N : TN \to N$ denotes the projection of a tangent vector onto its foot point, the above bundles are given by

$$\begin{aligned} T_f \operatorname{Imm}(M, N) &= \{h \in C^\infty(M, TN) : \pi_N \circ h = f\}, \\ \operatorname{Tan}_f &= \{Tf \circ X \mid X \in \mathfrak{X}(M)\}, \\ \operatorname{Nor}_f &= \{h \in T_f \operatorname{Imm}(M, N) \mid \forall k \in \operatorname{Tan}_f : \bar{g}(h, k) = 0\}. \end{aligned}$$

One specific example of a metric for which the splittings in Eqs. (2) and (3) coincide is the L^2 metric, which is also the simplest and in a way most natural metric on $\operatorname{Imm}(M, N)$. It is defined as

$$G_f^{L^2}(h, k) := \int_M \bar{g}(h, k) \operatorname{vol}(g).$$

Here, $h, k \in T_f \operatorname{Imm}(M, N)$ are tangent vectors at $f \in \operatorname{Imm}(M, N)$, $g = f^* \bar{g}$ denotes the pullback of the metric $\bar{g}$ along the immersion f and $\operatorname{vol}(g)$ denotes the associated volume form on M.