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A notion of nonpositive curvature for general metric spaces ☆

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ABSTRACT

We introduce a new definition of nonpositive curvature in metric spaces and study its relation to the existing notions of nonpositive curvature in comparison geometry. The main feature of our definition is that it applies to all metric spaces and does not rely on geodesics. Moreover, a scaled and a relaxed version of our definition are appropriate in discrete metric spaces, and are believed to be of interest in geometric data analysis.

1. Introduction

Originally, (sectional) curvature was conceived as a measure of the deviation of a surface, or—in Riemann's work—more generally a Riemannian manifold, from being flat. It was then found that curvature characterizes, and in turn can be characterized by the difference between the sum of the angles in a geodesic triangle and π, and also by the convergence or divergence rate of two geodesics emanating from a common point in different directions. In particular, the latter property is meaningful in spaces that are more general than Riemannian manifolds. One only needs a metric for which points can be connected by shortest curves, that is, geodesics. While for such spaces one cannot in general define a precise value of the curvature at a point, as in fact, curvature is a tensor and not a number, one could at least define curvature bounds in terms

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of the above mentioned properties. Perhaps the most important case is that of nonpositive curvature. There have been various such concepts developed, most notably those of Busemann and Alexandrov. A space that has nonpositive curvature in the sense of Alexandrov is also called a CAT(0) space, and such a space enjoys many nice geometric properties that are generalizations of those holding for (complete, simply connected) Riemannian manifolds of nonpositive sectional curvature.

In this paper, we take a further step towards an abstract definition of curvature inequalities. We introduce a new definition of nonpositive curvature (or more general curvature bounds) in metric spaces. Similarly to the definitions of Busemann and CAT(0) spaces, it is based on comparing triangles in the metric space in question with triangles in the Euclidean plane, but in contrast it does not require the space to be geodesic. Therefore, our definition extends in a meaningful and geometrically contentful manner to metric spaces that do not necessarily admit geodesics. In fact, a variant of our definition applies even to discrete metric spaces. We therefore see a lot of potential of our approach as a new tool in machine learning, extending the so-called topological data analysis, which is qualitative in nature, by some form of geometric data analysis that is more quantitative.

Let (X, d) be a metric space. A triple of points (a_1, a_2, a_3) in X is called a *triangle* and the points a_1, a_2, a_3 are called its *vertices*. For this triangle in (X, d), there exist points $\bar{a}_1, \bar{a}_2, \bar{a}_3 \in \mathbb{R}^2$ such that

$$d(a_i, a_j) = \|\bar{a}_i - \bar{a}_j\|, \quad \text{for every } i, j = 1, 2, 3,$$

where $\|\cdot\|$ stands for the Euclidean norm. The triple of points $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ is called a *comparison triangle* for the triangle (a_1, a_2, a_3), and it is unique up to isometries.

Given these two triangles, we define the functions

$$\rho_{(a_1,a_2,a_3)}(x) = \max_{i=1,2,3} d(x, a_i), \quad x \in X,$$

and,

$$\rho_{(\bar{a}_1,\bar{a}_2,\bar{a}_3)}(x) = \max_{i=1,2,3} \|x - \bar{a}_i\|, \quad x \in \mathbb{R}^2.$$

The numbers

$$r(a_1, a_2, a_3) := \inf_{x \in X} \rho_{(a_1,a_2,a_3)}(x) \quad \text{and} \quad r(\bar{a}_1, \bar{a}_2, \bar{a}_3) := \min_{x \in \mathbb{R}} \rho_{(\bar{a}_1,\bar{a}_2,\bar{a}_3)}(x)$$

are called the *circumradii* of the respective triangles. Next we can introduce our main definition.

Definition 1.1 *(Nonpositive curvature).* Let (X, d) be a metric space. We say that $\operatorname{Curv} X \le 0$ if, for each triangle (a_1, a_2, a_3) in X, we have

$$r(a_1, a_2, a_3) \le r(\bar{a}_1, \bar{a}_2, \bar{a}_3), \tag{1}$$

where $\bar{a}_i$ with $i = 1, 2, 3$ are the vertices of an associated comparison triangle.

As we shall see, our definition of nonpositive curvature is implied by the CAT(0) property, but not by nonpositive curvature in the sense of Busemann. In Riemannian manifolds, however, all of them are equivalent to global nonpositive *sectional* curvature. We also make a connection to the celebrated Kirszbraun extension theorem.

In order to appreciate the geometric content of our definition, let us assume that the infimum in (1) is attained, i.e., there exists some $m \in X$ with