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On infinitesimal Einstein deformations

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A B S T R A C T

We study infinitesimal Einstein deformations on compact flat manifolds and on product manifolds. Moreover, we prove refinements of results by Koiso and Bourguignon which yield obstructions on the existence of infinitesimal Einstein deformations under certain curvature conditions.

1. Introduction

Let M^n be a compact manifold of dimension $n \geq 3$ and let $\mathcal{M}$ be the set of smooth Riemannian metrics on it. Given an Einstein metric g, one may ask whether g is isolated in the set of Einstein structures, i.e. whether any other Einstein metric in a small neighborhood in $\mathcal{M}$ is homothetic to g.

To study this question, we consider infinitesimal Einstein deformations, that are symmetric 2-tensors h which are trace-free and divergence-free and satisfy the linearized Einstein equations

$$\Delta_E h := \nabla^* \nabla h - 2\mathring{R}h = 0.$$

Trace-free and divergence-free symmetric 2-tensors are also called TT-tensors. By ellipticity of the involved operator, the space of infinitesimal Einstein deformations is finite dimensional since M is compact. If g has no infinitesimal Einstein deformations, it is isolated in the space of Einstein structures. The converse is not true: The product metric on $S^2 \times \mathbb{C}P^{2n}$ is isolated although it has infinitesimal Einstein deformations [14].

Moreover, as is well-known, Einstein metrics of volume c are critical points of the Einstein–Hilbert action

$$S\colon \mathcal{M}_c \ni g \mapsto \int_M \mathrm{scal}_g \, dV_g$$

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[9]. Here, $\mathcal{M}_c$ is the set of metrics of volume c. Einstein metrics are always saddle points of the Einstein–Hilbert action but there is a notion of stability of Einstein manifolds which is as follows: We say that an Einstein manifold is stable, if $S''(h) \leq 0$ for all TT-tensors. We call it strictly stable if $S''(h) < 0$ for all nonzero TT-tensors. An Einstein manifold is strictly stable if and only if it is stable and does not admit infinitesimal Einstein deformations. This stability problem has been studied by Koiso [11,13,15], Dai, Wang and Wei [6,7] and in a recent paper by the author [16].

In this work, we study infinitesimal Einstein deformations on certain classes of manifolds. Throughout, any manifold M^n is compact and $n \geq 3$ unless the contrary is explicitly asserted. This work is organized as follows: In Section 3, we consider compact flat manifolds. We compute the dimension of infinitesimal Einstein deformations in terms of the holonomy of the manifold:

Theorem 1.1. *Let $(M = \mathbb{R}^n/G, g)$ be a Bieberbach manifold and let ρ be the canonical representation of the holonomy of G on $\mathbb{R}^n$. Let*

$$\rho \cong (\rho_1)^{i_1} \oplus \ldots \oplus (\rho_l)^{i_l}$$

be an irreducible decomposition of ρ. Then the dimension of the space of infinitesimal Einstein deformations is equal to

$$\dim(\ker(\Delta_E|_{TT}) = \sum_{j=1}^{l} \frac{i_j(i_j+1)}{2} - 1.$$

Here, TT denotes the space of TT-tensors. In Section 4, we consider products of Einstein spaces and we compute the kernel and the coindex of S'' restricted to TT-tensors on products of Einstein spaces. As a result of our discussion, we get

Theorem 1.2. *Let (M, g_1) be an Einstein manifold with positive Einstein constant μ and suppose, $2\mu \in \mathrm{spec}(\Delta)$. Then for any other Einstein manifold (N, g_2) with the same Einstein constant, the product manifold $(M \times N, g_1 + g_2)$ admits infinitesimal Einstein deformations.*

The dimension of the space of infinitesimal Einstein deformations is bounded from below by the multiplicity of the eigenvalue 2μ. By a result of Matsushima ([19], see also [2, Theorem 11.52]), a Kähler–Einstein metric with Einstein constant μ admits a holomorphic vector field if and only if 2μ is contained in the spectrum of the Laplacian. Therefore we obtain

Corollary 1.3. *Let (M, g_1) be a positive Kähler–Einstein manifold which admits a holomorphic vector field. Then for any other Einstein manifold (N, g_2) with the same Einstein constant, the product manifold $(M \times N, g_1 + g_2)$ admits infinitesimal Einstein deformations.*

This allows us to generate large families of Einstein manifolds which have infinitesimal Einstein deformations. In fact, all known Kähler–Einstein manifolds with $c_1 > 0$ admit holomorphic vector fields [2, Remark 12.101]. In Section 5, we refine the following stability criterions which are well-known from the literature:

Corollary 1.4. *(Bourguignon, unpublished.) Let (M, g) be an Einstein manifold such that the sectional curvature lies in the interval $(\frac{n-2}{3n}, 1]$. Then (M, g) is strictly stable.*

Corollary 1.5. *(See [11, Proposition 3.4].) Let (M, g) be an Einstein manifold with sectional curvature $K < 0$. Then (M, g) is strictly stable.*