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On the linearly independent vector fields on Grassmann manifolds

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ABSTRACT

In this paper are found $\theta(n)$ linearly independent vector fields on the Grassmann manifold $G_k(V)$ of k-planes in n-dimensional Euclidean vector space if k is odd number, where $\theta(n)$ is the maximal number of linearly independent vector fields on S^{n-1}, i.e. skewsymmetric anticommuting complex structures on $\mathbb{R}^n$.

1. Introduction

The well known paper of Adams [1] states that the maximal number $\theta(n)$ of linearly independent vector fields on the sphere S^{n-1} is given by

$$\theta(n) = 2^\beta + 8\alpha - 1$$

if $n = 2^{4\alpha+\beta} \cdot (2s+1)$, where $\alpha, s \in \mathbb{N}_0$, $\beta \in \{0, 1, 2, 3\}$. Indeed, the construction of such $\theta(n)$ vector fields on S^{n-1} was known much earlier [3], but Adams [1] proved that there do not exist more than $\theta(n)$ linearly independent vector fields on S^{n-1}. At the same time, about 50 years ago when the paper of Adams [1] was published, Clifford modules were introduced [2]. They are representations of the Clifford algebras and the use of them throws considerable light on the periodicity theorem for the stable orthogonal group. A fundamental result on Clifford modules is that the Morita equivalence class of a Clifford algebra, i.e. the equivalence class of the category of Clifford modules over it, depends only on the signature $p - q \pmod 8$, which is an algebraic form of Bott periodicity.

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In this section we give a brief explanation about the correspondence among the structure of $Cl(\mathbb{R}^k)$-module on $\mathbb{R}^n$, skewsymmetric anticommuting complex structures on $\mathbb{R}^n$ and the linearly independent vector fields on sphere S^{n-1}.

The Clifford algebra C_k is defined as free associative $\mathbb{R}$-algebra generated by 1 and $e_1, \cdots, e_k$, subject to the relations

$$e_i^2 = -1, \qquad e_i e_j + e_j e_i = 0 \quad \text{for } i \neq j.$$

These relations specify that we can get the set of words $\{e_{i_1} \cdots e_{i_s} \mid s \geq 0, i_1 < \cdots < i_s\}$ as a basis for C_k and hence $\dim C_k = 2^k$. The Clifford algebras C_k for $k = 0, 1, \cdots, 8$ are given by [2]

$$C_0 = \mathbb{R}, \qquad C_1 \cong \mathbb{C}, \qquad C_2 \cong \mathbb{H}, \qquad C_3 \cong \mathbb{H} \oplus \mathbb{H}, \qquad C_4 \cong \mathbb{H}(2),$$
$$C_5 \cong \mathbb{C}(4), \qquad C_6 \cong \mathbb{R}(8), \qquad C_7 \cong \mathbb{R}(8) \oplus \mathbb{R}(8), \qquad C_8 \cong \mathbb{R}(16).$$

Moreover, the Clifford algebras are periodic with period 8, in the sense that $C_{k+8} = C_k \otimes C_8 = C_k \otimes \mathbb{R}(16)$, whence if $C_k \cong F(m)$ then, $C_{k+8} \cong F(16m)$. If $n = 2^{4\alpha+\beta} \cdot (2s+1)$ ($\alpha, s \in \mathbb{N}_0$, $\beta \in \{0, 1, 2, 3\}$), and $m = 2^\alpha + 8\beta$, having in mind the structures of C_k there exist $m - 1 = \theta(n)$ automorphisms $e_1, \cdots, e_{m-1}$ on $\mathbb{R}^n$, such that $e_i^2 = -1$ and $e_i e_j + e_j e_i = 0$ for $i \neq j$, which are indeed anticommuting complex structures and further we will denote them by $J_1, \cdots, J_{m-1}$.

These $m - 1 = \theta(n)$ anticommuting complex structures induce the same number of non-vanishing vector fields on S^{n-1} in the following way. This number of linearly independent vector fields depends only on the even part of n. Let $J_0 = I$ and let G be the multiplicative finite subgroup of C_k of order 2^m generated by $\pm J_i$, $0 \leq i \leq m-1$. Further we choose a metric on $\mathbb{R}^n$ such that G preserves the metric. Using that $J_1, \cdots, J_{m-1}$ are orthogonal complex structures, they must be skewsymmetric and for each $\vec{v} \in S^{n-1}$ and $i \neq j$ we obtain

$$(J_i\vec{v}) \cdot (J_j\vec{v}) = \vec{v}^T J_i^T J_j \vec{v} = -\vec{v}^T J_j^T J_i \vec{v} = -(J_j\vec{v}) \cdot (J_i\vec{v}) = -(J_i\vec{v}) \cdot (J_j\vec{v}).$$

Thus $J_1\vec{v}, \cdots, J_{m-1}\vec{v}$ are mutually orthogonal tangent vectors and hence they are linearly independent.

If S is irreducible $Cl(\mathbb{R}^n)$ module, then the left multiplication

$$J_i := L_{e_i} : x \mapsto e_i \cdot x$$

defines anticommuting complex structures in S, which generate linearly independent vector fields in the unit sphere in S.

2. Preliminaries about the tangent spaces of the Grassmann manifolds

Before we present the main results in Section 3, here we give some preliminaries about Grassmann manifolds and their tangent bundles [4,6]. The Grassmann manifold $G_k(V)$ consists of k-planes ($k < n$) of n-dimensional Euclidean vector space V. The set $G_k(V)$ is a quotient of a subset of $V \times \cdots \times V$ consisting of linearly independent k-tuples of vectors with the subspace topology. The topology on $G_k(V)$ is just the quotient topology. It is a homogeneous space and the general linear group acts transitively on $G_k(V)$ with an isotropy group consisting of automorphisms preserving a given subspace U. The group of isometries $O(V)$ acts transitively and the isotropy group of U is $O(U^\perp) \times O(U)$, where $U^\perp$ is the orthogonal complement of U. The Grassmann manifold $G_k(V)$ around $U \in G_k(V)$ is locally modeled on the vector space $\mathrm{Hom}(U, U^\perp)$. Indeed, let $\mathcal{U}$ be an open subset of $G_k(V)$ consisting of all k-planes Z such that the orthogonal projection $p : V = U \oplus U^\perp \to U$ maps Z onto U, i.e. $\mathcal{U} = \{Z \in G_k(V) \mid Z \cap U^\perp = 0\}$. Then each $Z \in \mathcal{U}$ can be