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On vanishing theorems for Higgs bundles

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A R T I C L E I N F O

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A B S T R A C T

We introduce the notion of Hermitian Higgs bundle as a natural generalization of the notion of Hermitian vector bundle and we study some vanishing theorems concerning Hermitian Higgs bundles when the base manifold is a compact complex manifold. We show that a first vanishing result, proved for these objects when the base manifold was Kähler, also holds when the manifold is compact complex. From this fact and some basic properties of Hermitian Higgs bundles, we conclude several results. In particular we show that, in analogy to the classical case, there are vanishing theorems for invariant sections of tensor products of Higgs bundles. Then, we prove that a Higgs bundle admits no nonzero invariant sections if there is a condition of negativity on the greatest eigenvalue of the Hitchin–Simpson mean curvature. Finally, we prove that the invariant sections of certain tensor products of a weak Hermitian–Yang–Mills Higgs bundle are all parallel in the classical sense.

1. Introduction

As it is well known, in complex geometry one has some results on vanishing of holomorphic sections of a holomorphic vector bundle under certain negativity conditions on the Chern mean curvature of the bundle. These results, first proved by Bochner and Yano [17], have been used by Gauduchon [8] and Kobayashi [11] to study some properties of Hermitian vector bundles over compact complex manifolds. In particular, Kobayashi used some of these properties to prove one direction of the Hitchin–Kobayashi correspondence for holomorphic vector bundles over compact Kähler manifolds; namely, Kobayashi proved the polystability of such a bundle if the bundle was Hermitian–Einstein. The other direction of this correspondence has been proved by Donaldson [6,7] when the base manifold was a compact complex projective manifold, and by Uhlenbeck and Yau [18] when the manifold was compact Kähler. The Hitchin–Kobayashi correspondence plays an important role in Complex Geometry and is the subject of much active research, it has been studied in detail by Lübke and Teleman [12] in the case of holomorphic vector bundles when the base manifold is compact complex, and has been extended to coherent sheaves over compact Kähler manifolds by Bando and Siu [1].

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On the other hand, the notion of a Higgs bundle was introduced by Hitchin [10] and Simpson [14,15]. They used this concept to construct an extension of the Hitchin–Kobayashi correspondence. In particular, in [10] Hitchin used some vanishing theorems to prove that an irreducible Higgs bundle over a compact Riemann surface of zero degree is stable if and only if it satisfies the Hermitian–Yang–Mills condition. In [14], Simpson extended the result of Hitchin for Higgs bundles over Kähler manifolds of arbitrary dimension and degree, which is indeed the Hitchin–Kobayashi correspondence for Higgs bundles. Following the ideas of Bando and Siu [1], Biswas and Schumacher [2] proved this correspondence also for Higgs sheaves. Now, Bruzzo and Graña Otero [3] proved one of these vanishing results for Higgs bundles over compact Kähler manifolds, and they used it to proved that a Higgs bundle is semistable if it admits an approximate Hermitian–Yang–Mills metric. Finally, Seaman in [13] studied some particular examples of Higgs bundles, which arise in a natural way from bundles of holomorphic forms, and he proved some vanishing theorems for holomorphic forms on these bundles.

This article is organized as follows, in the first section we introduce the notion of a Hermitian Higgs bundle over a compact Hermitian manifold and we make some comments about the basic properties of these objects. In particular, we review the notion of invariant section of a Hermitian Higgs bundle. For Hermitian Higgs bundles we can apply the same operations that are commonly applied to Hermitian vector bundles, and hence make sense to study the same notions that are introduced in Complex Geometry. In this section we review the notion of Hitchin–Simpson curvature and we show that we get a formula for the corresponding mean curvature, which is indeed similar to the classical one. We also introduce the concept of weak Hermitian–Yang–Mills metric, as a natural generalization of the concept of weak Hermitian–Einstein metric introduced by Kobayashi [11]. In the final part of this section we review the classical Weitzenböck formula, a key result that can be used also for Hermitian Higgs bundles.

In the second section we prove some Bochner's vanishing theorems for Hermitian Higgs bundles over compact complex manifolds. In the first part we show that if the Hitchin–Simpson mean curvature of a Hermitian Higgs bundle is seminegative definite everywhere, every invariant section is parallel in the classical sense, i.e., it is parallel with respect to the Chern connection. If moreover, the Hitchin–Simpson mean curvature is negative definite at some point, then there are no nonzero invariant sections for such a bundle. This result has been proved in [3], when the base manifold is Kähler. Here we modify their proof to cover also the general case. Then we show that, in analogy to the classical case, there is also a vanishing theorem for tensor products of Hermitian Higgs bundles and from this result we get some corollaries. Next, we prove the main theorem of this article. Namely, we prove that if the eigenvalues of the Hitchin–Simpson mean curvature of a Hermitian Higgs bundle satisfy certain negativity condition, then such a bundle admits no nonzero invariant sections. This result is again an extension of a classical result for Hermitian vector bundles. Finally, we prove that if a Hermitian Higgs bundle satisfies the Hermitian–Yang–Mills condition, then on certain tensor products of this bundle, every invariant section is parallel in the classical sense.

2. Hermitian Higgs bundles

We start with some basic definitions. Let X be a compact complex manifold and denote by Ω^1_X the cotangent bundle to it. Following [10] and [14], a Higgs bundle $\mathfrak{E}$ over X is a holomorphic vector bundle E over X together with a map $\phi : E \to E \otimes \Omega^1_X$ such that $\phi \wedge \phi : E \to E \otimes \Omega^2_X$ vanishes. The map ϕ is called the Higgs field of $\mathfrak{E}$. On Higgs bundles we can apply the same operations that we apply to holomorphic bundles. In particular, the dual of a Higgs bundle is again a Higgs bundle, and tensor products of Higgs bundles are Higgs bundles. If $\mathfrak{E}$ is a Higgs bundle we denote its dual by $\mathfrak{E}^*$, and if $\mathfrak{E}_1$ and $\mathfrak{E}_2$ are Higgs bundles over X, we denote by $\mathfrak{E}_1 \otimes \mathfrak{E}_2$ its tensor product. For further details about these basic properties see for instance [4] and [5]. Now, in order to establish the vanishing theorems we need to use the notion of invariant section of a Higgs bundle. Following [3], we say that a section s of a Higgs bundle $\mathfrak{E}$ is ϕ-invariant if $\phi(s) = s \otimes \lambda$ for some holomorphic 1-form λ on X.