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On Matsumoto metrics of scale flag curvature ☆

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ABSTRACT

This paper contributes to the study of the Matsumoto metric $F = \frac{\alpha^2}{\alpha-\beta}$, where α is a Riemannian metric and β is a one form. It is shown that such a Matsumoto metric F is of scalar flag curvature if and only if F is projectively flat.

1. Introduction

In Finsler geometry, there are several important geometric quantities. In this paper, our main focus is on the flag curvature.

For a Finsler manifold (M, F), the flag curvature K at a point x is a function of tangent planes $P \subseteq T_xM$ and nonzero vectors $y \in P$. This quantity tells us how curved the space is. When F is Riemannian, K depends only on the tangent plane $P \subseteq T_xM$ and is just the sectional curvature in Riemannian geometry. A Finsler metric F is said to be of scalar flag curvature if the flag curvature K at a point x is independent of the tangent plane $P \subseteq T_xM$, that is, the flag curvature K is a scalar function on the slit tangent bundle $TM\backslash\{0\}$. It is known that every locally projectively flat Finsler metric is of scalar flag curvature. However, the converse is not true.

(α, β)-metrics form a special and important class of Finsler metrics which can be expressed in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on M and

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$\phi(s)$ is a C^∞ positive function satisfying (2.3) on some open interval. In particular, when $\phi(s) = \frac{1}{1-s}$, the Finsler metric $F = \frac{\alpha^2}{\alpha-\beta}$ is called a Matsumoto metric, which was first introduced by M. Matsumoto to study the time it takes to negotiate any given path on a hillside (cf. [1]). Recently, some researchers have studied Matsumoto metrics [8,9,18,17].

Randers metrics are the simplest (α,β)-metrics. Bao, etc., finally classifies Randers metrics of constant flag curvature by using the navigation method (see [2]). Further, Shen, etc., classifies Randers metrics of weakly isotropic flag curvature (see [12]). There are Randers metrics of scalar flag curvature which are not of weakly isotropic flag curvature or not locally projectively flat (see [3,11]). Besides, some relevant researches are refereed to [5,10,14], under additional conditions. So far, Randers metrics of scalar flag curvature are still unknown. Kropina metrics and m-Kropina metrics have been recently received more attention (see [13,15,16]).

Our main result concerns Matsumoto metrics of scalar flag curvature.

Theorem 1.1. *Let $F = \frac{\alpha^2}{\alpha-\beta}$ be a non-Riemannian Matsumoto metric on an n-dimensional manifold M, $n \geqslant 3$. Then F is of scalar flag curvature if and only if F is projectively flat, i.e., α is locally projectively flat and β is parallel with respect to α.*

Li obtains that an n ($\geqslant$3)-dimensional non-Riemannian Matsumoto metric is projectively flat if and only if α is locally projectively flat and β is parallel with respect to α (see [7]). It is known that α is locally projectively flat is equivalent to that α is of constant curvature. Hence, a Matsumoto metric, which is projectively flat (i.e., of scalar flag curvature), must be locally Minkowskian.

The content of this paper is arranged as follows. In Section 2 we introduce essential curvatures of Finsler metrics, as well as notations and conventions. And we give basic formulas for proofs of theorems in the following section. In Section 4 the characterization of scalar flag curvature is given under the assumption that the dual of β, with respect to α, is a constant Killing vector field. By using it, Theorem 1.1 is proved in Section 5.

2. Preliminaries

In this section, we give a brief description of several geometric quantities in Finsler geometry. See [4] in detail.

Let F be a Finsler metric on an n-dimensional smooth manifold M and $(x,y) = (x^i, y^i)$ the local coordinates on the tangent bundle TM. The geodesics of F are locally characterized by a system of second order ordinary differential equations

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x(t), \frac{dx(t)}{dt}\right) = 0,$$

where

$$G^i = \frac{1}{4}g^{ij}\left\{\left[F^2\right]_{x^k y^j} y^k - \left[F^2\right]_{x^j}\right\}.$$

G^i are called spray coefficients of F.

The Riemann curvature R (in local coordinates $R^i{}_k \frac{\partial}{\partial x^i} \otimes dx^k$) of F is defined by

$$R^i{}_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k} y^j + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

It is known that F is of scalar flag curvature if and only if, in a local coordinate system,