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The front of increasing concentric balls and cut locus in a surface

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ABSTRACT

Let M be a compact Riemannian 2-manifold without boundary and $o \in M$. We introduce an index for critical points of the distance function to o and establish the relation among the index, the properties of critical points and the Euler characteristic of M.

1. Introduction

Throughout this note let M be a complete smooth Riemannian surface without boundary and $o \in M$ a given point. Let $p \in M$ and $d_p : M \to \mathbb{R}$ the distance function to p which is induced from the Riemannian metric of M. Let $C(o)$ denote the cut locus of o and $d_o|_{C(o)}$ the distance function d_o restricted to $C(o)$. The cut locus $C(o)$ has locally the structure of a tree (see [6,7]). If M is compact, then $C(o)$ is pathwise connected.

We say that a path h contained in $C(o)$ is *d-constant* if h is not a single point and $d_p|_{C(o)}$ is constant on h. Any d-constant path in $C(o)$ is topologically either a segment or a circle (see Lemma 4). Moreover, M is homeomorphic to the projective surface $\mathbb{R}P^2(1)$ if h is topologically a circle (see Theorem 5).

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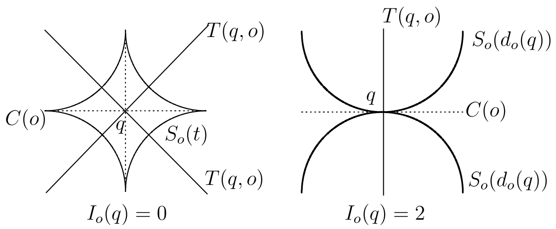
We write $x \sim y$ if x and y can be joined by a d-constant curve in $C(o)$. Then we have the quotient space $\widetilde{C}(o) = C(o)/\sim$, the natural projection $\pi : C(o) \to \widetilde{C}(o)$ and the function $\tilde{d}_o : \widetilde{C}(o) \to \mathbb{R}$ such that $d_o|_{C(o)} = \tilde{d}_o \circ \pi$. From the definition, $\tilde{d}_o$ is not constant in any open set in $\widetilde{C}(o)$.

Let T_xM denote the tangent space of M at $x \in M$. We say that $q \in C(o)$ is a *critical point* of d_o if, for any tangent vector $v \in T_qM$, there exists a minimizing geodesic segment $T(q,o)$ from q to o such that the angle of v with $T(q,o)$ at q is less than or equal to $\pi/2$. The points in a d-constant path in $C(o)$ are critical points of d_o. If there exist at least three minimizing geodesic segments from a critical point x of d_o to o such that x is not an endpoint of $C(o)$, then x is a branch cut point of $C(o)$. It is well known that the set of all branch critical points of $C(o)$ is discrete [6]. However, the set of other critical points may not be discrete, in general. In fact, in the projective surface $\mathbb{R}P^2(1)$, we see that all points in $C(o) = \{q \mid d(o,q) = \pi/2\}$ are critical points of d_o.

In order to classify the critical points $x \in C(o)$ in the intrinsic geometry, we define the index $I_o(x)$ (resp., $I_o(h)$) for the distance function d_o for all points $x \in M$ (resp., all d-constant path h in $C(o)$). Let $B_p(t) = \{x \in M \mid d_p(x) \le t\}$ and $B_h(t) = \{x \in M \mid d_h(x) \le t\}$ for a point $p \in M$, a d-constant path h and $t > 0$. Let $S_p(t) = \{x \in M \mid d_p(x) = t\}$. Let $I_{o,\varepsilon}(x)$ be the number of connected components of $B_x(\varepsilon) \setminus B_o(d_o(x))$. The index $I_{o,\varepsilon}(h)$ is similarly defined for a d-constant path h. We define

$$I_o(x) = \limsup_{\varepsilon \to 0+0} I_{o,\varepsilon}(x) \quad \text{and} \quad I_o(h) = \limsup_{\varepsilon \to 0+0} I_{o,\varepsilon}(h).$$

We will see that $I_{o,\varepsilon}(x)$ and $I_{o,\varepsilon}(h)$ converge as $\varepsilon \to 0+0$ if the set of extremal points of $\tilde{d}_o$ in $\widetilde{C}(o)$ is discrete (see Lemma 7, 8, 9, 10).



Actually, we will see that $q \in C(o)$ is a critical point of d_o if $I_o(q) = 2$ or $I_o(q) = 0$. Those points are local minimum or maximum points of $d_o|_{C(o)}$, respectively.

When $\widetilde{C}(o)$ is not a single point, it should be noted that if $q \in \widetilde{C}(o)$ is an endpoint which is a local minimum point of $\tilde{d}_o$, then $I_o(\pi^{-1}(q)) = 1$. Moreover, we will see that $\pi^{-1}(q) \subset C(o)$ behaves like a non-critical point. Let $\widetilde{C}(o)_e$ denote the set of all endpoints of $\widetilde{C}(o)$ where $\tilde{d}_o$ attains a local minimum if $\widetilde{C}(o)$ is not a single point. Otherwise, let $\widetilde{C}(o)_e = \emptyset$.

Let $\chi(X)$ denote the Euler characteristic of a subspace $X \subset M$.

Theorem 1. *Let M be a compact surface such that $C(o)$ does not contain a closed d-constant curve. Assume that the set of extremal points of $\tilde{d}_o$ in $\widetilde{C}(o)$ is discrete. If the numbers of local minimum and maximum points of $\tilde{d}_o$ in $\widetilde{C}(o) \setminus \widetilde{C}(o)_e$ are m and n, respectively, then $\chi(M) = 1 - m + n$.*

Here, an *extremal point* $\tilde{x} \in \widetilde{C}(o)$ of $\tilde{d}_o$ is by definition a local minimum or maximum point of $\tilde{d}_o$. We see in Example 12 that the set of extremal points of $\tilde{d}_o$ in $\widetilde{C}(o)$ is not discrete.

The idea is to think of the cut locus of o as the trace of points where the front of increasing geodesic balls centered at o clashes so as to break the smoothness. Two points hit on it, and so on, like the wave front does. This idea provides us with the classification of critical points of d_o.