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Total lightcone curvatures of spacelike submanifolds in Lorentz–Minkowski space

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ABSTRACT

We introduce the totally absolute lightcone curvature for a spacelike submanifold with general codimension and investigate global properties of this curvature. One of the consequences is that the Chern–Lashof type inequality holds. Then the notion of lightlike tightness is naturally induced.

1. Introduction

In this paper we consider global properties of spacelike submanifolds in Lorentz–Minkowski space. The study of the extrinsic differential geometry of submanifolds in Lorentz–Minkowski space is of interest in the special relativity theory. Moreover, it is a natural generalization of the extrinsic geometry of submanifolds in Euclidean space. In [9] the case of codimension two spacelike submanifolds has been considered. The normalized lightcone Gauss map was introduced which plays the similar role to the Gauss map of a hypersurface in the Euclidean space. For example, the Gauss–Bonnet type theorem holds for the corresponding Gauss–Kronecker curvature (cf., [9, Theorem 6.5]). Moreover, we recently discovered a new geometry on the hyperbolic space which is different from the Gauss–Bolyai–Lobachevskii geometry (i.e., the hyperbolic geometry) [1,2,6,8]. We call this new geometry the *horospherical geometry*. The horospherical Gauss map (or, the hyperbolic Gauss map) is one of the key notions in the horospherical geometry. We also showed that

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the Gauss–Bonnet type theorem holds for the horospherical Gauss–Kronecker curvature [8]. The notion of normalized lightcone Gauss maps unifies both the notion of Gauss maps in the Euclidean space and the notion of horospherical Gauss maps in the hyperbolic space.

In this paper we generalize the normalized lightcone Gauss map and the corresponding curvatures for general spacelike submanifolds in Lorentz–Minkowski space. If we try to develop this theory as a direct analogy to the Euclidean case, there exist several problems. The main problem is that the fiber of the unit normal bundle of a spacelike submanifold is a union of the pseudo-spheres which is not only non-compact but also non-connected. So, we cannot integrate the curvatures along the fiber at each point. Therefore, we cannot define the Lipschitz–Killing curvature analogous to the Euclidean case directly [5]. In order to avoid this problem, we arbitrary choose a future directed unit normal vector field along the submanifold and consider the pseudo-orthonormal space of this timelike vector on each fiber of the normal bundle. Then we obtain a spacelike codimension two unit normal sphere bundle in the normal bundle over the submanifold whose fiber is the Euclidean sphere. As a consequence, we define the normalized lightcone Lipschitz–Killing curvature and the total absolute lightcone curvature at each point. We remark that the values of these curvatures are not invariant under the Lorentzian motions. However, the flatness with respect to the curvature is an invariant property. We can show that the total absolute lightcone curvature is independent of the choice of the unit future directed timelike normal vector field (cf., Lemma 6.2). Although these curvatures are not Lorentzian invariant, we show that the Chern–Lashof type inequality holds for this curvature (cf., Section 7). In Section 8 we consider codimension two spacelike submanifolds. In this case the situation is different from the higher codimensional case. We have two different normalized lightcone Lipschitz–Killing (i.e., Gauss–Kronecker) curvatures at each point. The corresponding total absolute normalized lightcone Lipschitz–Killing (i.e., Gauss–Kronecker) curvatures are also different (cf., the remark after Theorem 8.3). However, we also have the Chern–Lashof type inequality for each total absolute Lipschitz–Killing (i.e., Gauss–Kronecker) curvature. Moreover, we consider the Willmore type integral (cf., [14, Theorem 7.2.2]) of the lightcone mean curvature for spacelike surface in Lorentz–Minkowski 4-space. Finally, we introduce the notion of the lightlike tightness which characterize the minimal value of the total absolute lightcone curvature. As a special case, we have the horo-spherical Chern–Lashof type inequality and horo-tight immersions in the hyperbolic space [1,2,13]. Motivated by those arguments, we can introduce the notion of several kinds of tightness and tautness depending on the causal characters which will be one of the subjects of a future program of the research.

2. Basic concepts in Lorentz–Minkowski space

We introduce in this section some basic notions on Lorentz–Minkowski $n+1$-space. For basic concepts and properties, see [12].

Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R}\ (i = 0, 1, \dots, n)\}$ be an $n+1$-dimensional Cartesian space. For any $\boldsymbol{x} = (x_0, x_1, \dots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo-scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i.$$

We call $(\mathbb{R}^{n+1}, \langle , \rangle)$ *Lorentz–Minkowski* $n+1$-*space* (or, simply *Minkowski* $n+1$-*space*). We write $\mathbb{R}^{n+1}_1$ instead of $(\mathbb{R}^{n+1}, \langle , \rangle)$. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is *spacelike, lightlike or timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$ respectively. The norm of the vector $\boldsymbol{x} \in \mathbb{R}^{n+1}_1$ is defined to be $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. We have the canonical projection $\pi : \mathbb{R}^{n+1}_1 \longrightarrow \mathbb{R}^n$ defined by $\pi(x_0, x_1, \dots, x_n) = (x_1, \dots, x_n)$. Here we identify $\{\boldsymbol{0}\} \times \mathbb{R}^n$ with $\mathbb{R}^n$ and it is considered as Euclidean n-space whose scalar product is induced from the pseudo scalar product $\langle , \rangle$. For a vector $\boldsymbol{v} \in \mathbb{R}^{n+1}_1$ and a real number c, we define a *hyperplane with pseudo-normal* $\boldsymbol{v}$ by