Differential Geometry and its Applications 34 (2014) 139–165

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Constructing reparameterization invariant metrics on spaces of plane curves

Martin Bauer [a,1], Martins Bruveris [b,2], Stephen Marsland [c,3], Peter W. Michor [a,4]

[a] *Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, A-1090 Wien, Austria*
[b] *Institut de mathématiques, EPFL, CH-1015 Lausanne, Switzerland*
[c] *School of Engineering and Advanced Technology, Massey University, Private Bag 11-222, Palmerston North, New Zealand*

ARTICLE INFO

Article history:
Received 27 January 2014
Available online 9 May 2014
Communicated by J. Slovák

MSC:
58B20
58D15
65D18

Keywords:
Curve matching
Elastic metric
Geodesic shooting
Reparameterization group
Riemannian shape analysis
Shape space

ABSTRACT

Metrics on shape spaces are used to describe deformations that take one shape to another, and to define a distance between shapes. We study a family of metrics on the space of curves, which includes several recently proposed metrics, for which the metrics are characterised by mappings into vector spaces where geodesics can be easily computed. This family consists of Sobolev-type Riemannian metrics of order one on the space $\operatorname{Imm}(S^1, \mathbb{R}^2)$ of parameterized plane curves and the quotient space $\operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$ of unparameterized curves. For the space of open parameterized curves we find an explicit formula for the geodesic distance and show that the sectional curvatures vanish on the space of parameterized open curves and are non-negative on the space of unparameterized open curves. For one particular metric we provide a numerical algorithm that computes geodesics between unparameterized, closed curves, making use of a constrained formulation that is implemented numerically using the RATTLE algorithm. We illustrate the algorithm with some numerical tests between shapes.

1. Introduction

The mathematical analysis of shape has been the focus of intense research interest in recent years, not least because of applications in image analysis and computer vision, where methods based on geodesic active contours or ‘snakes’ are used for segmentation, tracking and object recognition [31,32]. Another source of applications is biomedical image analysis, where the study and comparison of shapes form a large part of the field of computational anatomy [12,13].

E-mail addresses: bauer.martin@univie.ac.at (M. Bauer), martins.bruveris@epfl.ch (M. Bruveris), s.r.marsland@massey.ac.nz (S. Marsland), peter.michor@univie.ac.at (P.W. Michor).

[1] Supported by Austrian Science Fund grant P 24625.
[2] Supported by the Advanced ERC Grant FCCA.
[3] Supported by Royal Society of New Zealand Marsden Fund grant MAU0908.
[4] Supported by Austrian Science Fund grant P 21030-N13.

http://dx.doi.org/10.1016/j.difgeo.2014.04.008
0926-2245/

A key problem in shape analysis is to define a distance function between shapes that can measure similarity in a computationally feasible way and act as the basis for object classification. One way to arrive at such a distance function is to equip the space of shapes with a Riemannian metric, which allows the lengths of paths between shapes to be measured. The distance between two shapes can then be defined as the length of the shortest path connecting them.

For the purposes of this paper we consider shapes to be smooth plane curves (open or closed) modulo smooth reparameterizations. A slightly narrower definition would be to define a shape as the outline of a smooth, simply connected domain in the plane; this definition excludes objects like the figure of eight, which we allow in our definition. Mathematically, a shape is represented by a smooth curve $c : S^1 \to \mathbb{R}^2$. To curves c, d represent the same shape if there exists a reparameterization map $\varphi \in \operatorname{Diff}(S^1)$ (that is, the group of smooth, orientation preserving, invertible maps $\varphi : S^1 \to S^1$ from the circle onto itself) such that one curve is a reparameterization of the other, i.e., $c = d \circ \varphi$.

We will work with the class of regular or immersed curves c. A curve c is regular if it has a non-vanishing tangent, i.e., $c'(\theta) \neq 0$. The diffeomorphism group $\operatorname{Diff}(S^1)$ acts on the space of immersed curves

$$\operatorname{Imm}\left(S^1, \mathbb{R}^2\right) := \left\{c \in C^\infty\left(S^1, \mathbb{R}^2\right) \mid c'(\theta) \neq 0\right\},$$

from the right via $(\varphi, c) \mapsto c \circ \varphi$. Using this setting we can identify a shape with an equivalence class $[c]$, that is an element of the quotient space $\operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$. This quotient is the shape space of immersed closed curves modulo reparameterizations and will be denoted by $\mathcal{S}$. Similarly, one defines the space $\operatorname{Imm}([0, 2\pi], \mathbb{R}^2)$ of parameterized open curves and the space $\mathcal{S}_{\text{open}}$ of open shapes.

To arrive at a distance function on shape space requires two steps. First, we define a Riemannian metric (i.e., an inner product measuring the length of infinitesimal deformations of a curve) on the space of immersed curves and compute geodesics on this space. The deformations h, k are represented by vector fields along the curve c and the inner product, which depends on the curve, is denoted by $G_c(h, k)$. If the metric is invariant under the action of the reparameterization group $\operatorname{Diff}(S^1)$, then it induces a Riemannian metric on shape space $\mathcal{S}$, which in turn gives rise to the geodesic distance function. The second step is to find the right representatives c, d of the equivalence classes $[c]$ and $[d]$, such that the geodesic distance $\operatorname{dist}^{\operatorname{Imm}}(c, d)$ coincides with $\operatorname{dist}^{\mathcal{S}}([c], [d])$.

1.1. Shape metrics and related work

The simplest reparameterization invariant metric on $\operatorname{Imm}(S^1, \mathbb{R}^2)$ is the L^2-metric

$$G_c(h, k) = \int_{S^1} \langle h, k\rangle \, ds,$$

where we integrate over arc-length, $ds = |c'(\theta)| d\theta$. However, the geodesic distance induced by this metric vanishes, i.e., the distance between any two shapes is 0, which renders it unsuitable for shape analysis [23].

One way to overcome this is to add terms involving higher derivatives of h and k to the metric, such as:

$$G_c(h, k) = \int_{S^1} \langle h, k\rangle + A\langle D_s h, D_s k\rangle \, ds,$$

where $D_s h = \frac{1}{|c'|} h'$ denotes the arc-length derivative of h, and $A > 0$ is a constant. This leads to the class of Sobolev-type metrics, which were independently introduced in [9,25,32] and studied further in [4,28].

Another family of metrics, the almost local metrics [5,6,24], prevent the geodesic distance from vanishing by introducing a weight function in the integral. Examples of weight functions involving the curvature or length are $w(\theta) = 1 + A\kappa(\theta)^2$ and $w(\theta) = \ell(c)^{-1}$.