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Cubics and negative curvature

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Riemannian cubics are curves that generalise cubic polynomials to arbitrary Riemannian manifolds, in the same way that geodesics generalise straight lines. Considering that geodesics can be extended indefinitely in any complete manifold, we ask whether Riemannian cubics can also be extended indefinitely. We find that there are always exceptions in Riemannian manifolds with strictly negative sectional curvature. On the other hand, we show that Riemannian cubics can always be extended in complete locally symmetric Riemannian manifolds of non-negative curvature.

1. Introduction

The problem of interpolating a sequence of points in a Riemannian manifold M using a curve has led to the proposal of *Riemannian cubics*. Given a smooth map $x : [0, T] \to M$, and a vector field X defined along x, write $\frac{DX}{dt}$ for the covariant derivative of X along x. A Riemannian cubic is a critical point of the *total squared acceleration*

$$\int_0^T \left\langle \frac{D}{dt}\frac{dx}{dt}, \frac{D}{dt}\frac{dx}{dt} \right\rangle dt, \tag{1}$$

among those curves with specified values for $x(0)$, $x(T)$, $\frac{dx}{dt}(0)$, $\frac{dx}{dt}(T)$. The Euler–Lagrange equation is (see [8])

$$\frac{D^3}{dt^3}\frac{dx}{dt} = -R\left(\frac{D}{dt}\frac{dx}{dt}, \frac{dx}{dt}\right)\frac{dx}{dt} \tag{2}$$

where R is the Riemannian curvature tensor. Interpolation curves formed from piecewise Riemannian cubics were independently proposed for applications in computer graphics in [2], for statistics in [4] (although the paper's main concern is another interpolation method), and for robotic control in [8]. The integral (1) is one of the simplest functionals depending on second derivatives – we see Riemannian cubics as a prototype for the study of higher order variational problems.

Riemannian cubics are higher order geodesics, in the same relation to geodesics as cubic polynomials are to affine lines. The mathematics of cubics is known to be much richer than for geodesics, even concerning questions about the basic properties of these curves. In the present paper, motivated by the large body of work on the long term dynamics of geodesics (see for example [3,11]), we answer some simple questions about the long term dynamics of Riemannian cubics.

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We are interested in the solutions of (2) for values of t outside of the interval $[0, T]$ specified in the variational problem (1). In the rotation group SO(3) equipped with a bi-invariant metric, geodesics are projective lines, while the long term dynamics of (2) are complicated. (See [6] for the long term dynamics of typical Riemannian cubics in SO(3), or [5,7] for a special case about which more is known.)

Any compact semisimple Lie group admits a bi-invariant Riemannian metric, and the solutions of (2) can be extended to the whole real line. This was proven in [6] for SO(3) but the method there applies in the general case of a compact semisimple Lie group. As we will see in this paper, in any locally symmetric complete Riemannian manifold whose sectional curvature is everywhere non-negative, the Riemannian cubics can be extended to the real line. Surprisingly, the situation is completely different for spaces of negative sectional curvature.

In an arbitrary complete Riemannian manifold M with a specified point p, one defines the map $\exp_p : T_pM \to M$ which takes the vector $A \in T_pM$ to $x(1)$, where x is the geodesic with $x(0) = p$ and $\frac{dx}{dt}(0) = A$. If M is locally symmetric with non-negative sectional curvature, we can do something similar for Riemannian cubics: for a specified $p \in M$ and $A \in T_pM$ let $\exp_A : T_pM \oplus T_pM \to TM$ be the map which takes B, C to $\frac{dx}{dt}(1)$ where x is the cubic with $x(0) = p$, $\frac{dx}{dt}(0) = A$, $\frac{D}{dt}\frac{dx}{dt}(0) = B$, $\frac{D^2}{dt^2}\frac{dx}{dt}(0) = C$. On arbitrary M this map may only be defined on a subset of $T_pM \oplus T_pM$. We will show that if the sectional curvature is strictly negative, there exist initial conditions for which Eq. (2) does not have solutions for all $t \in \mathbb{R}$. Since cubics stay cubics after the reparametrisation $t \to \alpha t$ (where $\alpha \in \mathbb{R}$), there exist initial conditions for which the equations do not have solutions defined even on the interval $[0, 1]$.

Write $K(\sigma)$ for the sectional curvature in a tangent plane σ of some point in M. We say that M has *strictly negative sectional curvature* if there exists some $\lambda > 0$ such that for any $x \in M$ and any plane σ in T_xM, we have $K(\sigma) \leqslant -\lambda$.

The main results of the paper are as follows: In Section 3 we prove the main theorem that in any manifold of strictly negative curvature, initial conditions can be chosen for Riemannian cubics whose speed diverges to infinity in finite time; thus these cubics cannot be extended to $\mathbb{R}$. After that we restrict our attention to locally symmetric spaces. In Section 5 we give an example: we find initial conditions for a cubic in the hyperbolic plane for which $\langle \frac{dx}{dt}, \frac{dx}{dt} \rangle$ can be solved exactly; it diverges in finite time. In Section 6 we prove that in *locally symmetric spaces* of non-negative sectional curvature, Riemannian cubics can be extended to $\mathbb{R}$.

For some applications length may be incorporated into the functional. *Riemannian cubics in tension* are critical points of

$$\int_0^T \left\langle \frac{D}{dt}\frac{dx}{dt}, \frac{D}{dt}\frac{dx}{dt} \right\rangle + \tau \left\langle \frac{dx}{dt}, \frac{dx}{dt} \right\rangle dt$$

where τ is a positive real constant. The Euler–Lagrange equation is

$$\frac{D^3}{dt^3}\frac{dx}{dt} = -R\left(\frac{D}{dt}\frac{dx}{dt}, \frac{dx}{dt}\right)\frac{dx}{dt} + \tau \frac{D}{dt}\frac{dx}{dt}. \tag{3}$$

Cubics in tension were introduced in [13,14] (where they are called *elastic curves*). Their behaviour in a Lie group with a bi-invariant metric, and in particular in SO(3), is studied in [14,9,10]. The results of Sections 3 and 6 apply also to Riemannian cubics in tension.

2. Preliminary calculations

In this section we make some calculations that will be needed throughout the paper. Recall the following identities of the Riemannian curvature tensor R. For X, Y, Z, W tangent vectors at a point (see for example [12, Chapter 15]):

$$R(X,Y)Z = -R(Y,X)Z, \tag{4}$$

$$\langle R(X,Y)Z, W \rangle = -\langle R(X,Y)W, Z \rangle, \tag{5}$$

$$\langle R(X,Y)Z, W \rangle = \langle R(W,Z)Y, X \rangle. \tag{6}$$

Let p be a point in a Riemannian manifold M and let $X, Y \in T_pM$ be linearly independent. The *sectional curvature* in the plane $\sigma = \operatorname{span}\{X, Y\}$ is

$$K(\sigma) = \frac{\langle R(X,Y)Y, X \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2} \tag{7}$$

which, because of the symmetries of R, does not depend upon the choice of X and Y spanning σ.

Let $\tau \geqslant 0$. (We treat (2) as a special case of (3) with $\tau = 0$.) For any solution $x : (t_-, t_+) \to M$ of (3) and for non-negative integers i, j define

$$F_{ij} = \left\langle \frac{D^i}{dt^i}\frac{dx}{dt}, \frac{D^j}{dt^j}\frac{dx}{dt} \right\rangle.$$

Then we have