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On quasi-umbilical locally strongly convex homogeneous affine hypersurfaces ☆

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ABSTRACT

In this paper, by developing the techniques of F. Dillen and L. Vrancken in [6], we study quasi-umbilical locally strongly convex homogeneous unimodular-affine hypersurfaces. We will present a characterization of a certain subclass in all dimensions; finally, in dimension five, we will give a complete classification of all quasi-umbilical homogeneous unimodular-affine hypersurfaces.

1. Introduction

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional equiaffine space and M be a non-degenerate unimodular-affine hypersurface of $\mathbb{R}^{n+1}$. M is called locally homogeneous if for all points p and q of M, there exists a neighborhood U_p of p in M, and an equiaffine transformation A of $\mathbb{R}^{n+1}$, i.e., $A \in SL(n+1, \mathbb{R}^{n+1}) \ltimes \mathbb{R}^{n+1}$, such that $A(p) = q$ and $A(U_p) \subset M$. If $U_p = M$ for all p, M is called homogeneous.

Let S denote the affine shape operator of M, which is symmetric with respect to the affine metric. When M is locally strongly convex, S is diagonalizable. If S is a multiple of the identity, M is called an affine sphere. If S at each point has an eigenvalue λ with multiplicity exactly $n-1$, we call M *quasi-umbilical*.

While homogeneous affine surfaces have been classified in [14,15], the class of higher dimensional homogeneous affine hypersurfaces is very large, and one is far from a complete classification. For 3-dimensional locally strongly convex hypersurfaces, a complete classification of locally homogeneous affine hypersurfaces

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was achieved by Dillen and Vrancken [4–6]. When the affine metric is indefinite, M. Ooguri [16,17] further classified 3-dimensional quasi-umbilical locally homogeneous affine hypersurfaces in case that the hypersurfaces are not affine spheres. For higher dimensional locally homogeneous affine hypersurfaces, only partial results are known, see [5,6,10] for details.

In particular for locally strongly convex homogeneous affine spheres, Sasaki [18] reduces the classification of homogeneous hyperbolic affine spheres to that of homogeneous convex cones. We noticed that affine hypersurfaces with parallel cubic form are locally homogeneous affine spheres [7], and most recently, such homogeneous affine spheres have been completely classified in two cases, namely in the locally strongly convex case in [8], and in the Lorentzian case in [9].

We recall some known results in more detail, namely: For quasi-umbilical locally strongly convex homogeneous affine hypersurfaces, Dillen and Vrancken [6] showed that one eigenvalue of its affine shape operator must be zero. Moreover, the following results are known:

Theorem 1.1. *(See [5].) Let M be a locally strongly convex locally homogeneous affine hypersurface with* $\operatorname{rank}(S) = 1$ *in $\mathbb{R}^{n+1}$. Then M is affine equivalent to the convex part of the hypersurface satisfying the equation*

$$\left(z - \frac{1}{2}\sum_{i=1}^{p} x_i^2\right)^{p+2} \left(w - \frac{1}{2}\sum_{j=1}^{q} y_j^2\right)^{q+2} = 1,$$

where $p + q = n - 1$ and $(x_1, \ldots, x_p, y_1, \ldots, y_q, z, w)$ are the coordinates of $\mathbb{R}^{n+1}$.

Theorem 1.2. *(See [6].) Let M^3 be a quasi-umbilical locally strongly convex locally homogeneous affine hypersurface in $\mathbb{R}^4$. Then M is affine equivalent to the convex part of one of the following hypersurfaces*:

(i) $(y - \frac{1}{2}(x^2 + z^2))^4 w^2 = 1,$
(ii) $(y - \frac{1}{2}x^2)^3 (z - \frac{1}{2}w^2)^3 = 1,$
(iii) $(y - \frac{1}{2}x^2)^3 z^2 w^2 = 1,$
(iv) $(y - \frac{1}{2}x^2 - \frac{1}{2}w^2/z)^4 z^3 = 1,$

where (x, y, z, w) are the coordinates of $\mathbb{R}^4$.

Theorem 1.3. *(See [6].) Let M^4 be a quasi-umbilical locally strongly convex locally homogeneous affine hypersurface in $\mathbb{R}^5$. Then M is affine equivalent to the convex part of one of the following hypersurfaces*:

(i) $(y - \frac{1}{2}(x^2 + z^2 + u^2))^5 w^2 = 1,$
(ii) $(y - \frac{1}{2}(x^2 + u^2))^4 (z - \frac{1}{2}w^2)^3 = 1,$
(iii) $(y - \frac{1}{2}x^2)^3 z^2 w^2 u^2 = 1,$
(iv) $(y - \frac{1}{2}x^2 - \frac{1}{2}(w^2/z + u^2/z))^5 z^4 = 1,$
(v) $(y - \frac{1}{2}x^2 - \frac{1}{2}w^2/z)^4 z^3 u^2 = 1,$
(vi) $(y - \frac{1}{2}x^2)(z^2 - (w^2 + u^2)) = 1,$

where (x, y, z, w, u) are the coordinates of $\mathbb{R}^5$.

In this paper, we continue to study quasi-umbilical locally strongly convex homogeneous affine hypersurfaces. By developing the techniques of [6], we will prove Theorem 3.1 which, for all dimensions, characterizes a special quasi-umbilical homogeneous affine hypersurface. Further, as one of the main results, we obtain the following classification theorem for dimension $n = 5$.