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Conformal positive mass theorems for asymptotically flat manifolds with inner boundary

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A R T I C L E   I N F O

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Inspired by Witten's insightful spinor proof of the positive mass theorem, in this paper, we use the spinor method to derive higher dimensional type conformal positive mass theorems on asymptotically flat spin manifolds with inner boundary, which states that under a condition about the plus (minus) relation between the scalar curvatures of the original and the conformal metrics in addition with some boundary condition, we will get the associated positivity of their ADM masses. The rigidity part of the plus part is used in the proof of black hole uniqueness theorems. They are related with quasi-local mass and the spectrum of Dirac operator.

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1. Introduction

In General Relativity, ADM mass [1] is one of the most important global invariants for asymptotically flat manifolds. The celebrated "Positive Mass conjecture" is about the positivity of the ADM mass which shows that the energy-momentum vector is a forward pointing timelike vector in the general spacetime, and the rigidity statement that the ADM mass is zero if and only it is flat. The proofs were given first by R. Schoen and S.T. Yau [21], then by E. Witten [19,24] using the spinor method.

In the application of the positive mass theorem, taking a suitable conformal rescaling of the metric is of an important tool to many aspects of mathematics and physics [14]. Using spinors, this applies to Herzlich's proof [9] of a Penrose-type inequality, which is about the lower bound of the ADM mass contributed by a collection of black holes. Later, W. Simon and M. Mars constructed two conformal positive mass theorems [22,15] which can be used to prove the uniqueness of static black holes. Inspired by his result, we want to establish higher dimensional theorems of this type under proper conditions to maintain the positivity of mass and generalize it to manifolds with inner boundary, which will involve some relation with

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Brown–York mass in all dimension and the Dirac spectrum on the noncompact manifolds with boundary. The main goal of our paper is to use the spinor method to handle the case with inner boundary (not necessarily apparent horizon) in all dimension. Otherwise, in Section 6 we prove that the uniqueness of static black holes theorems can be derived only when the boundary is round sphere. In this sense, considering the boundary effect is an interesting issue to modify the black hole uniqueness problems.

In [22], W. Simon proved a conformal positive mass theorem and illustrate some physical uses of his theorem, here is what he proved:

Theorem 1.1. *(See [22].) Let (N, g) and (N, g') be asymptotically flat Riemannian 3-manifolds with compact interior and finite mass, such that g and g' are related via $g' = \phi^4 g$ with a $C^{2,\alpha}$-function $\phi > 0$. Assume further that $S \pm \phi^4 S' \geqslant 0$ (S denotes the scalar curvature). Then the corresponding masses satisfy $m \pm m' \geqslant 0$. Moreover, in the case of the minus sign, equality holds iff g and g' are isometric, the plus sign, iff (N, g) and (N, g') are flat Euclidean spaces.*

For the minus sign of this theorem, we can generalize it to higher dimension easily, when considering with inner boundary it will involve the Brown–York mass even in higher dimensions.

For the plus sign of this theorem, it is well known that such conformal positive mass theorem is a valid tool in the proof of the uniqueness of static charged black hole by A.K.M. Masood-ul-Alam [17]. While in the string theory, the inner dimensions will add to the space dimensions of spacetime, so considering higher dimensions is of importance to physical interest. Some higher dimensional conformal positive mass theorems are used to solve the uniqueness of charged black p-branes by Gibbons, Ida and Shiromizu [8], where they established a higher dimensional theorem, which is a rather simple proof based on [22,15] without using spinors. But in fact in using higher dimension (bigger than seven) positive mass theorem, spin condition seems to be unavoidable for the moment.

Theorem 1.2. *(See Appendix in [8].) Let (M, g) be an asymptotically flat spin n-manifold, $g' = e^f g$ such that (M, g') is also asymptotically flat. If $\beta e^f S_{g'} + S_g \geqslant 0$ where $\beta > 0$ is a constant, then $m_g + \beta m_{g'} \geqslant 0$. Moreover equality holds iff (M, g) and (M, g') are flat Euclidean spaces.*

In their proof, they take a new metric $\bar{g} = e^{\frac{2\beta f}{1+\beta}} g$, it can be shown that under $\beta e^f S_{g'} + S_g \geqslant 0$ they will get $S_{\bar{g}} \geqslant 0$, while by the relation $m_{\bar{g}} = \frac{m_g + \beta m_{g'}}{1+\beta}$, using the positive mass theorem for $(M, \bar{g})$, they finish their theorem.

Now we turn to our case, all the manifold we consider in this paper would be smooth and the inner boundary is compact. In our notation, we set $e^f = \phi^4$.

When we consider the manifold with inner boundary, it is obviously that their method cannot be used, because their representation of mass doesn't involve the boundary effect. In fact, in the use of their theorem, they patch the manifolds along the Killing horizon to avoid the boundary effect, the problem actually appears when the boundary is out of the Killing horizon. So using Witten's spinor representation of mass should be a good choice to extend Simon's result to manifolds with boundary, and this is what we want to do, we will establish conformal positive mass theorems in this case, these are Theorem 3.1 and Theorem 4.2. The statement will be seen in the paragraph, we can state that taking away the boundary term we get analogue conformal positive mass theorem as Theorem 1.2. In particular, taking the coupling constant $\beta = 1$, we have

Theorem 1.3. *Let (M, g) be an asymptotically flat spin n-manifold, $g' = e^f g$ such that (M, g') is also asymptotically flat. If $e^f S_{g'} - S_g \geqslant 0$, then $m_g - m_{g'} \geqslant 0$. If $\frac{1}{n-2} e^f S_{g'} + S_g \geqslant 0$, then $m_g + m_{g'} \geqslant 0$. Moreover, in the case of the minus sign, equality holds iff g and g' are isometric, the plus sign, iff (M, g) and (M, g') are flat Euclidean spaces.*

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