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Morse–Bott inequalities in the presence of a compact Lie group action and applications ☆

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ABSTRACT

In this paper, we obtain Morse–Bott inequalities in the presence of a compact Lie group action via Bismut–Lebeau's analytic localization techniques. As an application, we obtain Morse–Bott inequalities on compact manifold with nonempty boundary by applying the generalized Morse–Bott inequalities to the doubling manifold.

1. Introduction

In his influential work [24], Witten sketched analytic proofs of the degenerate Morse inequalities of Bott [8] for Morse functions whose critical submanifolds are nondegenerate in the sense of Bott. Rigorous proofs were given by Bismut [3], by using heat kernel methods, and later by Helffer and Sjöstrand [16], by means of semiclassical analysis. Braverman and Farber [12] provided another proof using the Witten deformation techniques suggested by Bismut [3].

Concerning the standard Morse inequalities (i.e., for Morse functions with isolated critical points), an analytic proof is given by Zhang [28, Chapter 5], in the spirit of the analytic localization techniques developed by Bismut–Lebeau [5, §8–9]. In fact, [28, Chapters 5–6] serves as a nice exposition of the Witten deformation in [6,7]. We also refer the readers to [6,7] for more details on applications of the Witten deformation. Following the ideas in [28], we give here a proof of Morse–Bott inequalities by similar techniques.

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Let us mention the related papers [12,14,15]. In [12,14], Braverman, Farber and Silantyev used Witten deformation techniques to study the Novikov number associated to closed differential 1-forms nondegenerate in the sense of Bott and Kirwan, respectively. In this way, they obtained Novikov-type inequalities associated to a closed differential 1-form. When the closed differential form is exact, these inequalities turn to Morse inequalities.

In [15], Feng and Guo established Novikov's type inequalities associated to vector fields instead of closed differential forms under a natural assumption on the zero-set of the vector field. These inequalities were actually first suggested by Novikov in the appendix to [20] and the results of this appendix were extended and explained in more details by Shubin in [21]. Feng and Guo [15] generalized the results of Novikov and Shubin.

In this paper, by using techniques from Bismut–Lebeau [5, §8–9] and Zhang [28] we work out Morse–Bott inequalities under the action of a compact Lie group. More precisely, we obtain Morse inequalities for the multiplicities of the irreducible representations of the group. To our knowledge this result is new even in the case of isolated critical points. Compared to [15], where Bismut–Lebeau's analytic localization techniques are applied along the lines of [28], we can choose here the geometrical data near the singular points as simple as possible (cf. [3, §2]), due to the equivariant Morse's Lemma [23]. As an application of the result described above, we get Morse–Bott inequalities for manifolds with nonempty boundary by passing to the doubling manifold. Thus, we extend the result from [27] to the most general situation.

In this paper we consider the usual de Rham cohomology of a manifold with the induced group action. Let us note that there exists a rich literature devoted to equivariant Morse inequalities regarding the equivariant cohomology introduced by Atiyah–Bott, see [1,4,9,19,24–26]. See also [11,13] for equivariant Novikov inequalities case.

Let M be a smooth m-dimensional closed and connected manifold. A smooth function $f : M \to \mathbb{R}$ is called Morse–Bott if the critical points of f form a union of disjoint connected submanifolds $Y_1, \dots, Y_{r'}$ such that for every $x \in Y_i$ the Hessian of f is nondegenerate on all subspaces of T_xM intersecting T_xY_i transversally (cf. [8]). Assume now that a compact Lie group G acts smoothly on M and that f is a G-invariant Morse–Bott function. One verifies directly that the index of the Hessian of f is constant on any orbit $G \cdot Y_i$. Set $\{B_1, \dots, B_r\} = \{G \cdot Y_1, \dots, G \cdot Y_{r'}\}$, $r \leqslant r'$, where $B_1, \dots, B_r$ are pointwise disjoint orbits. Then B_i is a G-invariant submanifold of M. For $1 \leqslant i \leqslant r$, let n_i be the dimension of the submanifold B_i and n_i^- be the index of the Hessian of f on B_i.

Using the equivariant Morse's Lemma [23], we embed each critical submanifold B_i in a G-invariant tubular neighborhood $(h, N_i^- \oplus N_i^+)$ of B_i such that h equivariantly embeds $N_i^- \oplus N_i^+$ into M. Moreover, there is an open G-invariant neighborhood $\mathcal{B}_i$ of B_i in $N_i^- \oplus N_i^+$ such that if $Z = (Z^-, Z^+) \in \mathcal{B}_i$, then

$$f \circ h\big(Z^-, Z^+\big) = c - \frac{|Z^-|^2}{2} + \frac{|Z^+|^2}{2}, \tag{1.1}$$

where c denotes the constant $f|_{B_i}$. The rank of N_i^- is n_i^-, while that of N_i^+ is $m - n_i - n_i^-$. Let $o(N_i^-)$ denote the orientation bundle of N_i^-. We call n_i^- the index of B_i in M.

In the sequel, we will often omit the subscript i in B_i, n_i, n_i^-, i.e., n denotes the dimension of the critical submanifold B and n^- is the index. Denote by $o(N^-)$ the orientation bundle of N^- over B.

Let W_1, W_2 be two finite-dimensional G-representations. A morphism between W_1 and W_2 is a linear map which is G-equivariant. Let $\mathrm{Hom}_G(W_1, W_2)$ denote the set of all morphism between W_1 and W_2. If E_1, E_2 are two finite-dimensional representations of G, then we say that

$$E_1 \leqslant E_2 \tag{1.2}$$