Differential Geometry and its Applications 31 (2013) 472–485

Contents lists available at SciVerse ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

The Ricci flow in a class of solvmanifolds ☆

Romina M. Arroyo

FaMAF & CIEM, Universidad Nacional de Córdoba, Córdoba, Argentina

ARTICLE INFO

Article history:
Received 4 December 2012
Available online 10 May 2013
Communicated by D.V. Alekseevsky

MSC:
53C44
53C30
22E25
53C25

ABSTRACT

In this paper, we study the Ricci flow of solvmanifolds whose Lie algebra has an abelian ideal of codimension one, by using the bracket flow. We prove that solutions to the Ricci flow are immortal, the ω-limit of bracket flow solutions is a single point, and that for any sequence of times there exists a subsequence in which the Ricci flow converges, in the pointed topology, to a manifold which is locally isometric to a flat manifold. We give a functional which is non-increasing along a normalized bracket flow that will allow us to prove that given a sequence of times, one can extract a subsequence converging to an algebraic soliton, and to determine which of these limits are flat. Finally, we use these results to prove that if a Lie group in this class admits a Riemannian metric of negative sectional curvature, then the curvature of any Ricci flow solution will become negative in finite time.

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1. Introduction

The Ricci flow is an evolution equation for a curve of Riemannian metrics on a manifold. In recent years, the Ricci flow has proven to be a very important tool. Many strong results, not only in Riemannian geometry, have been proven by using this equation. The objective of this paper is to study the Ricci flow for solvmanifolds whose Lie algebra has an abelian ideal of codimension one and get similar results to those obtained by J. Lauret in [8] in the case of nilmanifolds.

Let (G, g) be a solvmanifold, i.e. a simply connected solvable Lie group G endowed with a left-invariant metric g. Assume that the Lie algebra of G has an abelian ideal of codimension one. Consider the Ricci flow starting at g, that is,

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}\big(g(t)\big), \quad g(0) = g.$$

The solution $g(t)$ is a left-invariant metric for all t, thus each $g(t)$ is determined by an inner product on the Lie algebra. We will follow the approach in [10] to study the evolution of these metrics by varying Lie brackets instead of inner products.

More precisely, let μ be a Lie bracket on $\mathbb{R}^{n+1}$ with an abelian ideal of codimension one. We may assume that μ is determined by $A = \mathrm{ad}_\mu(e_0)|_{\mathbb{R}^n} \in \mathfrak{gl}_n(\mathbb{R})$, where $\mathbb{R}^{n+1} = \mathbb{R}e_0 \oplus \mathbb{R}^n$ and $\mathbb{R}^n$ is the abelian ideal, and so it will be denoted by μ_A. Each μ_A determines a Riemannian manifold (G_{μ_A}, g_{μ_A}), where G_{μ_A} is the simply connected Lie group with Lie algebra $(\mathbb{R}^{n+1}, \mu_A)$ and g_{μ_A} is the left-invariant metric determined by $\langle\cdot,\cdot\rangle$, the canonical inner product on $\mathbb{R}^{n+1}$. Every solvmanifold whose Lie algebra has an abelian ideal of codimension one is isometric to some μ_A (see Section 2). By [10, Theorem 3.3], the Ricci flow solution is given by $g(t) = \varphi(t)^* g_{\mu(t)}$, where $\mu(t)$ is a family of Lie brackets solving an ODE called the bracket flow, and $\varphi(t) : G \to G_{\mu(t)}$ is the Lie group isomorphism with derivative $h(t) : (\mathbb{R}^{n+1}, \mu) \to (\mathbb{R}^{n+1}, \mu(t))$, and $h(t)$ satisfies

☆ This research was supported by a fellowship from CONICET and grants from CONICET, FONCYT and SeCyT (Universidad Nacional de Córdoba).
E-mail address: arroyo@famaf.unc.edu.ar.

0926-2245/$ – see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.difgeo.2013.04.002

$$\frac{d}{dt}h = -h\,\mathrm{Ric}\big(\langle\cdot,\cdot\rangle_t\big), \qquad \frac{d}{dt}h = -\mathrm{Ric}_{\mu(t)}\,h, \quad h(0) = I.$$

In our case, we see that $\mu(t) = \mu_{A(t)}$, where $A(t) \in \mathfrak{gl}_n(\mathbb{R})$ is the solution to the following ODE,

$$\frac{d}{dt}A = -\operatorname{tr}\big(S(A)^2\big)A + \frac{1}{2}\big[A, \big[A, A^t\big]\big] - \frac{1}{2}\operatorname{tr}(A)\big[A, A^t\big], \quad A(0) = A,$$

and then we study the evolution of the matrix A. The main results in this paper can be summarized as follows:

- The Ricci flow solution $g(t)$ is defined for all $t \in (T_-, \infty)$, where $-\infty < T_- < 0$, and if $\operatorname{tr}(A^2) \geqslant 0$, then $g(t)$ is a Type-III solution (see Proposition 3.4 and Proposition 3.14).
- The scaling-invariant functional $\frac{\|[A(t),A(t)^t]\|}{\|A(t)\|^2}$ is strictly decreasing unless μ_A is an algebraic soliton, in which case it is constant (see Lemma 3.6). This happens precisely when A is either normal or nilpotent of a special kind (see Proposition 3.3).
- For any sequence $t_k \to \infty$, there exists a subsequence of $(G_{\mu_{A(t_k)}}, g_{\mu_{A(t_k)}})$ which converges in the pointed topology to a flat manifold, up to local isometry (see Corollary 3.11).
- If $\operatorname{tr}(A) = 0$ (i.e. G_{μ_A} unimodular), then $B(t) = \frac{A(t)}{\|A(t)\|}$ converges to a matrix B_∞, as $t \to \infty$ (see Lemma 4.1 and Remark 4.2).
- For any sequence $t_k \to \infty$, there exists a subsequence of $(G_{\mu_{B(t_k)}}, g_{\mu_{B(t_k)}})$ which converges in the pointed topology to $(G_{\mu_{B_\infty}}, g_{\mu_{B_\infty}})$ (up to local isometry), which is an algebraic soliton. In addition, $(G_{\mu_{B_\infty}}, g_{\mu_{B_\infty}})$ is non-flat, unless every eigenvalue of A is purely imaginary (see Theorem 5.2).
- If G_{μ_A} admits a negatively curved left-invariant metric, then there exists $t_0 > 0$ such that $g(t)$ is negatively curved for all $t \geqslant t_0$ (see Theorem 6.5). This is not true in general for solvmanifolds (see Example 6.6).

2. Preliminaries

2.1. The Ricci flow

Let (M, g) be a Riemannian manifold. The Ricci flow starting at (M, g) is the following partial differential equation:

$$\frac{\partial}{\partial t}g(t) = -2\,\mathrm{Rc}\big(g(t)\big), \quad g(0) = g, \tag{1}$$

where $g(t)$ is a curve of Riemannian metrics on M and $\mathrm{Rc}(g(t))$ the Ricci tensor of the metric $g(t)$.

A complete Riemannian metric g on a differentiable manifold M is a Ricci soliton if its Ricci tensor satisfies

$$\mathrm{Rc}(g) = cg + L_X g, \quad \text{for some } c \in \mathbb{R},\ X \in \chi(M) \text{ complete},$$

where $\chi(M)$ denotes the space of differentiable vector fields on M and L_X the usual Lie derivative in the direction of the field X.

Equivalently, Ricci solitons are precisely the metrics that evolve along the Ricci flow only by the action of diffeomorphisms and scaling (i.e. $g(t) = c(t)\varphi(t)^*g$), giving geometries that are equivalent to the starting point, for all time t (see [2] for more information about Ricci solitons).

Definition 2.1. A Ricci flow solution $g(t)$ is said to be of Type-III if it is defined for $t \in [0, \infty)$ and there exists $C \in \mathbb{R}$ such that

$$\big\|\mathrm{Rm}\big(g(t)\big)\big\| \leqslant \frac{C}{t}, \quad \forall t \in (0, \infty),$$

where $\mathrm{Rm}(g(t))$ is the Riemann curvature tensor of the metric $g(t)$.

2.2. Varying Lie brackets

We fix $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, with $\langle\cdot,\cdot\rangle$ an inner product on $\mathbb{R}^n$ and we define

$$\mathfrak{L}_n = \big\{\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n\colon\ \mu \text{ is bilinear, skew-symmetric and satisfies Jacobi}\big\},$$

$$\mathcal{N}_n = \{\mu \in \mathfrak{L}_n\colon\ \mu \text{ is nilpotent}\},$$

and $\mathrm{ad}_\mu : \mathbb{R}^n \to \mathbb{R}^n$ the adjoint representation of $\mu \in \mathfrak{L}_n$ (i.e. $\mathrm{ad}_\mu(x)(y) = \mu(x, y)$).

Then, $\mathrm{GL}_n(\mathbb{R})$ acts on $\mathfrak{L}_n$ by

$$h.\mu(X, Y) = h\mu\big(h^{-1}X, h^{-1}Y\big), \quad X, Y \in \mathbb{R}^n,\ h \in \mathrm{GL}_n(\mathbb{R}),\ \mu \in \mathfrak{L}_n. \tag{2}$$

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