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Generalized Cheeger–Gromoll metrics and the Hopf map

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ABSTRACT

We show that there exists a family of Riemannian metrics on the tangent bundle of a two-sphere, which induces metrics of constant curvature on its unit tangent bundle. In other words, given such a metric on the tangent bundle of a two-sphere, the Hopf map is identified with a Riemannian submersion from the universal covering space of the unit tangent bundle, equipped with the induced metric, onto the two-sphere. A hyperbolic counterpart dealing with the tangent bundle of a hyperbolic plane is also presented.

1. Introduction

One of the most studied maps in Differential Geometry is the Hopf map $H : \mathbb{S}^3 \to \mathbb{C}\mathrm{P}^1$ from the unit three-sphere $\mathbb{S}^3 \subset \mathbb{C}^2$ onto the complex projective line $\mathbb{C}\mathrm{P}^1 = \mathbb{C} \cup \{\infty\}$, defined for $z = (z_1, z_2) \in \mathbb{S}^3$ by

$$H(z) = \begin{cases} z_1/z_2 & \text{if } z_2 \neq 0, \\ \infty & \text{if } z_2 = 0. \end{cases}$$

Composed with the inverse stereographic projection $p^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{(0,0,1)\} \subset \mathbb{R}^3$ given by

$$p^{-1}(\zeta) = \left(\frac{2\,\mathrm{Re}\,\zeta}{|\zeta|^2 + 1}, \frac{2\,\mathrm{Im}\,\zeta}{|\zeta|^2 + 1}, \frac{|\zeta|^2 - 1}{|\zeta|^2 + 1} \right), \quad \zeta \in \mathbb{C},$$

it can be regarded as a map $H : \mathbb{S}^3 \to \mathbb{S}^2$ sending

$$z = (z_1, z_2) \mapsto \big(2\,\mathrm{Re}\, z_1 \bar{z}_2, 2\,\mathrm{Im}\, z_1 \bar{z}_2, |z_1|^2 - |z_2|^2\big), \tag{1.1}$$

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which, if we choose the two-sphere $\mathbb{S}^2$ to be of radius 1/2, becomes a Riemannian submersion, relative to the canonical metric on each sphere.

As is well known, the Hopf map is closely linked to the unit tangent bundle $T^1\mathbb{S}^2 \to \mathbb{S}^2$ of the two-sphere. Indeed, the total space $T^1\mathbb{S}^2$ is diffeomorphic to the real projective three-space $\mathbb{R}\mathrm{P}^3$, and the Hopf map $H : \mathbb{S}^3 \to \mathbb{S}^2$ is nothing else than the canonical projection from the universal covering space of $T^1\mathbb{S}^2$ onto $\mathbb{S}^2$. This shows that a Riemannian metric of constant positive curvature exists on $T^1\mathbb{S}^2$, inherited from the canonical metric on $\mathbb{S}^3$.

Then it is a pertinent question whether this constant curvature metric on $T^1\mathbb{S}^2$ is induced from some "natural" Riemannian metric defined on the "ambient" total space $T\mathbb{S}^2$ of the tangent bundle $T\mathbb{S}^2 \to \mathbb{S}^2$ of $\mathbb{S}^2$, when one regards the total space of the unit tangent bundle $T^1\mathbb{S}^2$ as a hypersurface of $T\mathbb{S}^2$. This question also arises when the three-sphere $\mathbb{S}^3$ is equipped with one of the Berger metrics, that is, when a homothety is applied on the fibres.

The aim of this paper is to give affirmative answers, using generalized Cheeger–Gromoll metrics $h_{m,r}$ defined in [1] (see Section 3.3 for the precise definition of $h_{m,r}$), that there is a two-parameter family of Riemannian metrics on the tangent bundle of $\mathbb{S}^2$, which induces desired metrics for both questions. Namely, we prove the following

Theorem 1.1. *Let $\mathbb{S}^n(c)$ be the n-sphere of constant curvature $c > 0$, and denote by $T\mathbb{S}^n(c)$ (resp. $T^1\mathbb{S}^n(c)$) its tangent (resp. unit tangent) bundle. Let $F : \mathbb{S}^3(c/4) \to T^1\mathbb{S}^2(c)$ be the covering map defined by (2.8).*

(1) *Then F induces an isometry from the projective three-space $(\mathbb{R}\mathrm{P}^3(c/4), g_{\mathrm{can}})$ of constant curvature $c/4$ to $T^1\mathbb{S}^2(c)$, equipped with the metric induced from the generalized Cheeger–Gromoll metric $h_{m,r}$ on $T\mathbb{S}^2(c)$, where $m = \log_2 c$ and $r \geqslant 0$.*
(2) *Similarly, when $\mathbb{S}^3$ is equipped with a Berger metric g_ϵ defined by (3.10), F induces an isometry from $(\mathbb{R}\mathrm{P}^3, g_\epsilon)$ to $(T^1\mathbb{S}^2(4), h_{m,r})$, for $m = \log_2 \epsilon^2 + 2$ and $r \geqslant 0$.*

In particular, we see from Theorem 1.1(1) that any three-sphere of constant positive curvature is isometrically immersed into the total space of the tangent bundle of a two-sphere, equipped with a generalized Cheeger–Gromoll metric. A hyperbolic counterpart of this is also true. Namely, any anti-de Sitter three-space of constant negative curvature is isometrically immersed into the total space of the tangent bundle of a hyperbolic plane, equipped with an indefinite generalized Cheeger–Gromoll metric. More precisely, we prove

Theorem 1.2. *Let $H_1^3(c)$ be the anti-de Sitter three-space of constant curvature $-c < 0$. Let $T\mathbb{H}^2(c)$ (resp. $T^1\mathbb{H}^2(c)$) be the tangent (resp. unit tangent) bundle of the hyperbolic plane $\mathbb{H}^2(c)$ of constant curvature $-c < 0$, and endow $T\mathbb{H}^2(c)$ with the indefinite generalized Cheeger–Gromoll metric $h_{m,r}$ defined by (4.14). Then the covering map $F : H_1^3(c/4) \to T^1\mathbb{H}^2(c)$ defined by (4.8) is an isometric immersion from $H_1^3(c/4)$ to $T^1\mathbb{H}^2(c)$, equipped with the metric induced from $h_{m,r}$, where $m = \log_2 c$ and $r \geqslant 0$.*

The paper is organized as follows. In Section 2 we describe the Hopf map $\mathbb{S}^3(c/4) \to \mathbb{S}^2(c)$ in terms of the natural identification of the three-sphere $\mathbb{S}^3(c/4)$ and the unit tangent bundle $T^1\mathbb{S}^2(c)$ with Lie groups $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$, respectively. Then, using these descriptions, we prove Theorem 1.1 in Section 3. For this end, we compute the differential of the covering map $F : \mathbb{S}^3(c/4) \to T^1\mathbb{S}^2(c)$ and find explicitly a suitable induced metric on $T^1\mathbb{S}^2(c)$ making F to be isometric. An alternative proof of Theorem 1.1, based on our previous knowledge of the curvature of generalized Cheeger–Gromoll metrics, is presented in Remark 3.3.

In Section 4 we prove a hyperbolic counterpart of Theorem 1.1(1). Namely, we define the hyperbolic Hopf map $H_1^3(c/4) \to \mathbb{H}^2(c)$ for the hyperbolic plane, and extend the notion of generalized Cheeger–Gromoll metrics to admit indefinite ones. Then we prove Theorem 1.2 by the same method as in Section 3, namely, by identifying the anti-de Sitter three-space $H_1^3(c/4)$ and the unit tangent bundle $T^1\mathbb{H}^2(c)$ with Lie groups $\mathrm{SU}(1,1)$ and $\mathrm{SO}^+(1,2)$, respectively.

2. Hopf map

To fix our notation and conventions, we first review how one can identify the Hopf map $H : \mathbb{S}^3 \to \mathbb{S}^2$ with the canonical projection from the universal covering space of the unit tangent bundle $T^1\mathbb{S}^2$ onto the 2-sphere $\mathbb{S}^2$.

To begin with, recall that the unit 3-sphere

$$\mathbb{S}^3 = \left\{ \left(x^1, x^2, x^3, x^4\right) \in \mathbb{R}^4 \;\middle|\; \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 + \left(x^4\right)^2 = 1 \right\}$$

is diffeomorphic to the special unitary group

$$\begin{aligned} \mathrm{SU}(2) &= \left\{ A \in \mathrm{GL}(2, \mathbb{C}) \;\middle|\; {}^t\bar{A}A = \mathrm{Id},\ \det A = 1 \right\} \\ &= \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \;\middle|\; a, b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\} \end{aligned}$$

under the map