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On the stability of minimal immersions into Finsler manifolds ☆

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ABSTRACT

In this paper, we derive the second variation formulas of volume for minimal immersions into Finsler manifolds and apply them to study the stability of minimal submanifolds. Then we prove that all minimal graphs in Minkowski $(n+1)$-space are stable. Furthermore, we obtain a Bernstein type theorem in Minkowski $(n+1)$-space by improving Bernstein type theorems in Euclidean space and give an example of unstable minimal surface in Minkowski 3-space.

1. Preliminaries

Theories on harmonic maps and minimal submanifolds have always been important research subjects in global differential geometry, and many significant results have been obtained in the last few decades. More recently, studies on harmonic maps and minimal submanifolds in Finsler geometry have also made some progress [1–6]. Using the Holmes–Thompson volume form, X.H. Mo introduced the notion of harmonic maps from a Finsler manifold to a Riemannian manifold in 2001 [3]. Later, Y.B. Shen and Y. Zhang derived the first and the second variation formulas of energy function for a nondegenerate map between Finsler manifolds [4]. The stability of harmonic maps was studied in [6]. In 2006, the first author and Y.B. Shen investigated the minimal immersions in Finsler manifolds, gave the first variation formulas of volumes [5] and proved some Bernstein type theorems for minimal graphs [9].

It is well known that though a minimal immersion is just an isometric harmonic map, their stabilities are different. The purpose of this paper is to study the second variation of volume and the stability of minimal submanifolds in Finsler geometry. In Section 2, we review some related definitions and formulas. In Section 3, we derive the second variation formulas of minimal immersions into Finsler manifolds (Theorem 1). As its application, we consider the stability of minimal immersions into Minkowski $(n+1)$-spaces. In Section 4, by simplifying the second variation formula, we give a necessary and sufficient condition for a minimal hypersurface in Minkowski $(n+1)$-space to be stable (Proposition 4.1) and prove that all minimal graphs in Minkowski space are stable (Theorem 2). In the last section, we obtain a Bernstein type theorem in Minkowski space (Theorem 4) by improving some Bernstein type theorems in Euclidean space (Theorem 3). Moreover, we

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show that a helicoidal surface is minimal not only in Euclidean 3-space but also in (α,β)-Minkowski space $(\tilde{V}^3, \tilde{\alpha}\phi(\frac{\tilde{\beta}}{\tilde{\alpha}}))$ (Example 1). By these facts, we give an example of unstable minimal surface in (α,β)-Minkowski spaces (Example 2).

2. Volume forms and minimal immersions

Let (M,F) be an n-dimensional smooth Finsler manifold. The natural projection $\pi : TM \to M$ gives rise to the pull-back bundle π^*TM and its dual π^*T^*M. Let (x,y) be a point of TM with $x \in M$, $y \in T_xM$, and (x^i, y^i) be the local coordinates on TM with $y = y^i\partial/\partial x^i$. We shall work on $TM \setminus \{0\}$ and rigidly use only objects that are invariant under positive rescaling in y, so that one may view them as objects on the projective sphere bundle SM using homogeneous coordinates. The following quantities

$$g_{ij} := \frac{1}{2}\big[F^2\big]_{y^iy^j}, \qquad A_{ijk} = \frac{F}{2}\left[\frac{1}{2}F^2\right]_{y^iy^jy^k}, \qquad \eta_i = g^{jk}A_{ijk}, \tag{2.1}$$

are called the *fundamental tensor*, the *Cartan tensor* and the *Cartan form* respectively. Here and from now on, $[F]_{y^i}$, $[F]_{y^iy^j}$ mean $\frac{\partial F}{\partial y^i}$, $\frac{\partial^2 F}{\partial y^i \partial y^j}$, etc., and we shall use the following convention of index ranges unless otherwise stated:

$$1 \leqslant i, j, \ldots \leqslant n; \qquad n+1 \leqslant a, b, \ldots \leqslant 2n-1; \qquad 1 \leqslant \alpha, \beta, \ldots \leqslant m(>n).$$

The simplest Finsler manifolds are Minkowski spaces, on which the metric function F is independent of x.

In π^*T^*M there is a global section $\omega = [F]_{y^i}\,dx^i$, called the *Hilbert form*, whose dual is $l = l^i\frac{\partial}{\partial x^i}$, $l^i = y^i/F$, called the *distinguished field*. Each fibre of π^*T^*M has a positively oriented orthonormal coframe $\{\omega^i\}$ with $\omega^n = \omega$. Expand ω^i as $v^i_j dx^j$, whereby the stipulated orientation implies that $\det(v^i_j) = \sqrt{\det(g_{ij})}$. Set

$$\omega^{n+i} = v^i_j \delta y^j, \qquad \delta y^i = \frac{1}{F}\big(dy^i + N^i_j\,dx^j\big), \qquad \frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} - N^k_j\frac{\partial}{\partial y^k}. \tag{2.2}$$

The collection $\{\omega^i, \omega^{n+i}\}$ forms an orthonormal basis on $T^*(TM\setminus\{0\})$ with respect to the Sasaki-type metric $g_{ij}\,dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$. The pull-back of the Sasaki-type metric from $TM \setminus \{0\}$ to SM is a Riemannian metric

$$\hat{g} = g_{ij}\,dx^i \otimes dx^j + \delta_{ab}\omega^a \otimes \omega^b. \tag{2.3}$$

Thus, the volume element dV_{SM} of SM with the metric $\hat{g}$ is given by

$$dV_{SM} = \omega^1 \wedge \cdots \wedge \omega^{2n-1} = \Omega\,d\tau \wedge dx, \tag{2.4}$$

where

$$\Omega := \det\left(\frac{g_{ij}}{F}\right), \qquad dx = dx^1 \wedge \cdots \wedge dx^n, \tag{2.5}$$

$$d\tau := \sum_{i=1}^{n}(-1)^{i-1}y^i dy^1 \wedge \cdots \wedge \widehat{dy^i} \wedge \cdots \wedge dy^n. \tag{2.6}$$

The *volume form* of a Finsler n-manifold (M,F) is defined by

$$dV_M := \sigma(x)\,dx, \qquad \sigma(x) := \frac{1}{c_{n-1}}\int_{S_xM}\Omega\,d\tau, \tag{2.7}$$

where c_{n-1} denotes the volume of the unit Euclidean $(n-1)$-sphere S^{n-1}, $S_xM = \{[y] \mid y \in T_xM\}$. It is well known that there exists uniquely the Chern connection ∇ on π^*TM with $\nabla\frac{\partial}{\partial x^j} = \omega^i_j\frac{\partial}{\partial x^i}$ and $\omega^i_j = \Gamma^i_{jk}\,dx^k$, satisfying

$$\begin{aligned} &d\big(dx^i\big) - dx^j \wedge \omega^i_j = 0, \\ &dg_{ij} - g_{ik}\omega^k_j - g_{jk}\omega^k_i = 2A_{ijk}\delta y^k. \end{aligned} \tag{2.8}$$

The curvature 2-forms of the Chern connection ∇ are

$$d\omega^i_j - \omega^k_j \wedge \omega^i_k = \Omega^i_j := \frac{1}{2}R_j{}^i{}_{kl}\,dx^k \wedge dx^l + P_j{}^i{}_{kl}\,dx^k \wedge \delta y^l, \tag{2.9}$$

where $R_j{}^i{}_{kl} = -R_j{}^i{}_{lk}$ and $P_j{}^i{}_{kl} = P_k{}^i{}_{jl}$ are called the *hh-curvature* and the *hv-curvature* respectively. The *Riemannian curvature tensor* and the *Landsberg curvature tensor* are defined by

$$R^i{}_j := R_s{}^i{}_{jk}l^sl^k, \qquad L^i{}_{jk} := -l^sP_s{}^i{}_{jk}, \tag{2.10}$$