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Regge's Einstein–Hilbert functional on the double tetrahedron ☆

Daniel Champion, David Glickenstein *, Andrea Young

University of Arizona, Tucson, AZ 85721, United States

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ABSTRACT

The double tetrahedron is the triangulation of the three-sphere gotten by gluing together two congruent tetrahedra along their boundaries. As a piecewise flat manifold, its geometry is determined by its six edge lengths, giving a notion of a metric on the double tetrahedron. We study notions of Einstein metrics, constant scalar curvature metrics, and the Yamabe problem on the double tetrahedron, with some reference to the possibilities on a general piecewise flat manifold. The main tool is analysis of Regge's Einstein–Hilbert functional, a piecewise flat analogue of the Einstein–Hilbert (or total scalar curvature) functional on Riemannian manifolds. We study the Einstein–Hilbert–Regge functional on the space of metrics and on discrete conformal classes of metrics.

1. Introduction

It is well known that Ricci-flat metrics on closed Riemannian manifolds of dimension at least three are critical points of the Einstein–Hilbert functional

$$\mathcal{EH}(M,g)=\int\limits_M R_g\,dV_g,$$

where R_g and dV_g are the scalar curvature and volume form for the closed Riemannian manifold (M,g). Since there are topological restrictions to being Ricci-flat (e.g., the Cheeger–Gromoll splitting theorem [10]), one may restrict to the subset of Riemannian manifolds with volume equal to 1 so that critical points of the constrained problem are Einstein manifolds. Equivalently, one can consider a normalized Einstein–Hilbert functional

$$\mathcal{NEH}\big(M^n,g\big)=\frac{\int_M R_g\,dV_g}{(\int_M dV_g)^{(n-2)/n}},$$

whose critical points are Einstein manifolds. Einstein manifolds are of interest because the Einstein metric is, in some sense, a most symmetric or "best" geometry for the manifold. In trying to prove a classification theorem such as Thurston's

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* Corresponding author.
E-mail addresses: champion@math.arizona.edu (D. Champion), glickenstein@math.arizona.edu (D. Glickenstein), ayoung@math.arizona.edu (A. Young).

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geometrization conjecture, one may try to find a best geometry by trying to optimize a geometric functional such as $\mathcal{EH}$ and by studying both convergence and degenerations to try to capture all possible "best" geometries (see [2]).

Related to the study of $\mathcal{EH}$ and Einstein manifolds is the well-known Yamabe problem, which asks whether one can find a constant scalar curvature metric within a conformal class or, equivalently, if one can find a critical point for $\mathcal{NEH}$ restricted to a conformal class. The solution was completed by R. Schoen, based on important contributions from Yamabe, Trudinger, and Aubin (see [20] for an overview and complete proof).

In assigning geometry to a topological manifold, an alternative to the Riemannian approach is that of piecewise flat geometry. A piecewise flat space is a triangulation together with edge lengths that determine a Euclidean geometry on each simplex in the triangulation. In 1961, T. Regge [23] suggested a functional defined on piecewise flat manifolds which is analogous to $\mathcal{EH}$. We call this functional the Einstein–Hilbert–Regge functional and denote it as $\mathcal{EHR}$. Study of this functional as an action for general relativity has led to a wide array of work on Regge calculus and lattice gravity (for a survey, see [16]). It was later shown that $\mathcal{EHR}$ and $\mathcal{EH}$ are related in the sense that appropriately finer piecewise flat triangulations which converge to a Riemannian manifold lead to convergence of the functionals. In fact, it was proven that the associated curvature measures converge [11]. Thus $\mathcal{EHR}$ is a discretization of $\mathcal{EH}$, and could potentially be used to approximate $\mathcal{EH}$. Such an approach is an alternative to discretizing the Einstein equations themselves. By discretizing the functional instead of its Euler–Lagrange equation, we hope to produce an approximation of the Euler–Lagrange equation whose behavior mimics that of the smooth case. This approach has been applied in a number of contexts, such as computer graphics, computational mechanics, and computational dynamics, and it is a primary focus of the fields of discrete differential geometry and discrete exterior calculus (see, e.g., [6,12,13,22]).

In addition, we can use a definition of conformal class in [15] to formulate a discrete version of the Yamabe problem. However, this does not allow us to reformulate the functional in the same way as in the smooth setting, which allows $\mathcal{NEH}$ to be rewritten in a relatively simply way as a function of the conformal factor. Instead, we are forced to work entirely with variation formulas for curvature.

The purpose of this paper is to consider the Einstein–Hilbert–Regge functional on the simplest possible triangulation of a three-manifold without boundary: the double tetrahedron. Even on this small triangulation, the behavior of the $\mathcal{EHR}$ functional is rich and complex. In particular, on the double tetrahedron we do not have a complete answer to the uniqueness of Einstein metrics, a complete understanding of the Yamabe problem, or a calculation of the Yamabe invariant. An extension of this work has been completed in [9]. In that paper, constant scalar curvature metrics are constructed on larger triangulations of $\mathbb{S}^3$, namely, boundary complexes of cyclic polytopes.

In Section 2, we give some background on piecewise flat manifolds, and we define the $\mathcal{EHR}$ functional and two normalized versions of it. Additionally, we describe the geometry of the double tetrahedron, and we set the notation for the remainder of the paper. In Section 3, we consider length variations of piecewise flat metrics on the double tetrahedron. Critical points of the normalized $\mathcal{EHR}$ functionals are geometrically significant and yield definitions of Einstein metrics in the piecewise flat setting. We study the convexity of the functionals at these points. In Section 4, we discuss discrete conformal variations of a piecewise flat metric as described in [15] (following [21,24,27]). The critical points of the normalized $\mathcal{EHR}$ functionals with respect to a conformal variation give rise to notions of constant scalar curvature. On the double tetrahedron, we are able to provide a partial classification of such metrics and are able to show existence in every conformal class. Additionally, we study the convexity of the curvature functionals at Einstein metrics. Finally, in Section 5 we discuss the Yamabe invariant on both the double tetrahedron and on general piecewise flat manifolds.

2. Background and notation

In this section we will secure notation for the rest of the paper. Most of the notation follows [15]. We will also provide the necessary background on piecewise flat manifolds, and we will define the double tetrahedron.

2.1. Geometry of the tetrahedron

Consider a Euclidean tetrahedron determined by four vertices numbered $1, 2, 3, 4$. The tetrahedron has six edge lengths, and we denote the length of the edge between vertices i and j by ℓ_{ij}. Since edge lengths arise from a nondegenerate tetrahedron, they satisfy a particular condition.

Definition 2.1. Consider the matrix A:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \ell_{12}^2 & \ell_{13}^2 & \ell_{14}^2 \\ 1 & \ell_{12}^2 & 0 & \ell_{23}^2 & \ell_{24}^2 \\ 1 & \ell_{13}^2 & \ell_{23}^2 & 0 & \ell_{34}^2 \\ 1 & \ell_{14}^2 & \ell_{24}^2 & \ell_{34}^2 & 0 \end{bmatrix}.$$

Let $\mathrm{CM}_3 = \det(A)$.