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On the geometry of connections with totally skew-symmetric torsion on manifolds with additional tensor structures and indefinite metrics ☆

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ABSTRACT

This paper is a survey of results obtained by the authors on the geometry of connections with totally skew-symmetric torsion on the following manifolds: almost complex manifolds with Norden metric, almost contact manifolds with B-metric and almost hypercomplex manifolds with Hermitian and anti-Hermitian metric.

0. Introduction

In Hermitian geometry there is a strong interest in the connections preserving the metric and the almost complex structure whose torsion is totally skew-symmetric [23,24,21,1,8,9,2,3]. Such connections are called *KT-connections* (or *Bismut connections*). They find widespread application in mathematics as well as in theoretic physics. For instance, it is proved a local index theorem for non-Kähler manifolds by KT-connection in [1] and the same connection is applied in string theory in [21]. According to [8], on any Hermitian manifold, there exists a unique KT-connection. In [3] all almost contact, almost Hermitian and G_2-structures admitting a KT-connection are described.

In this work we provide a survey of our investigations into connections with totally skew-symmetric torsion on almost complex manifolds with Norden metric, almost contact manifolds with B-metric and almost hypercomplex manifolds with Hermitian and anti-Hermitian metric.

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In Section 1 we consider an almost complex manifold with Norden metric (i.e. a neutral metric g with respect to which the almost complex structure J is an anti-isometry). On such a manifold we study a natural connection (i.e. a linear connection ∇' preserving J and g) and having totally skew-symmetric torsion. We prove that ∇' exists only when the manifold belongs to the unique basic class with non-integrable structure J. This is the class $\mathcal{W}_3$ of quasi-Kähler manifolds with Norden metric. We establish conditions for the corresponding curvature tensor to be Kählerian as well as conditions ∇' to have a parallel torsion. We construct a relevant example on a 4-dimensional Lie group.

In Section 2 we consider an almost contact manifold with B-metric which is the odd-dimensional analogue of an almost complex manifold with Norden metric. On such a manifold we introduce the so-called φKT-connection having totally skew-symmetric torsion and preserving the almost contact structure and the metric. We establish the class of the manifolds where this connection exists. We construct such a connection and study its geometry. We establish conditions for the corresponding curvature tensor to be of φ-Kähler type as well as conditions for the connection to have a parallel torsion. We construct an example on a 5-dimensional Lie group where the φKT-connection has a parallel torsion.

In Section 3 we consider an almost hypercomplex manifold with Hermitian and anti-Hermitian metric. This metric is a neutral metric which is Hermitian with respect to the first almost complex structure and an anti-Hermitian (i.e. a Norden) metric with respect to the other two almost complex structures. On such a manifold we introduce the so-called pHKT-connection having totally skew-symmetric torsion and preserving the almost hypercomplex structure and the metric. We establish the class of the manifolds where this connection exists. We study the unique pHKT-connection D on a nearly Kähler manifold with respect to the first almost complex structure. We establish that this connection coincides with the known KT-connection on nearly Kähler manifolds and therefore it has a parallel torsion. We prove the equivalence of the conditions D be strong, flat and with a parallel torsion with respect to the Levi-Civita connection.

1. Almost complex manifold with Norden metric

Let (M, J, g) be a $2n$-dimensional *almost complex manifold with Norden metric*, i.e. M is a differentiable manifold with an almost complex structure J and a pseudo-Riemannian metric g such that

$$J^2x = -x, \qquad g(Jx, Jy) = -g(x, y)$$

for arbitrary x, y of the algebra $\mathfrak{X}(M)$ on the smooth vector fields on M. Further x, y, z, w will stand for arbitrary elements of $\mathfrak{X}(M)$.

The associated metric $\tilde{g}$ of g on M is defined by $\tilde{g}(x, y) = g(x, Jy)$. Both metrics are necessarily of signature (n, n). The manifold $(M, J, \tilde{g})$ is also an almost complex manifold with Norden metric.

A classification of the almost complex manifolds with Norden metric is given in [4]. This classification is made with respect to the tensor F of type $(0, 3)$ defined by $F(x, y, z) = g((\nabla_x J)y, z)$, where ∇ is the Levi-Civita connection of g. The tensor F has the following properties

$$F(x, y, z) = F(x, z, y) = F(x, Jy, Jz), \qquad F(x, Jy, z) = -F(x, y, Jz). \tag{1}$$

The basic classes are $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$. Their intersection is the class $\mathcal{W}_0$ of the Kählerian-type manifolds, determined by $\mathcal{W}_0$: $F(x, y, z) = 0 \Leftrightarrow \nabla J = 0$.

The class $\mathcal{W}_3$ of the *quasi-Kähler manifolds with Norden metric* is determined by the condition

$$\mathcal{W}_3: \quad F(x, y, z) + F(y, z, x) + F(z, x, y) = 0. \tag{2}$$

This is the only class of the basic classes $\mathcal{W}_1$, $\mathcal{W}_2$ and $\mathcal{W}_3$, where each manifold (which is not a Kähler-type manifold) has a non-integrable almost complex structure J, i.e. the Nijenhuis tensor N, determined by $N(x, y) = (\nabla_x J)Jy - (\nabla_y J)Jx + (\nabla_{Jx} J)y - (\nabla_{Jy} J)x$ is non-zero.

The components of the inverse matrix of g are denoted by g^{ij} with respect to a basis $\{e_i\}$ of the tangent space T_pM of M at a point $p \in M$.

The *square norm of* ∇J is defined by $\|\nabla J\|^2 = g^{ij}g^{ks}g((\nabla_{e_i} J)e_k, (\nabla_{e_j} J)e_s)$.

Definition 1. (See [19].) An almost complex manifold with Norden metric and $\|\nabla J\|^2 = 0$ is called an *isotropic-Kähler manifold*.

1.1. KT-connection

Let ∇' be a linear connection on an almost complex manifold with Norden metric (M, J, g). If T is the torsion tensor of ∇', i.e. $T(x, y) = \nabla'_x y - \nabla'_y x - [x, y]$, then the corresponding tensor of type $(0, 3)$ is determined by $T(x, y, z) = g(T(x, y), z)$.

Definition 2. (See [5].) A linear connection ∇' preserving the almost complex structure J and the Norden metric g, i.e. $\nabla' J = \nabla' g = 0$, is called a *natural connection* on (M, J, g).