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Higher order variational origin of the Dixon's system and its relation to the quasi-classical 'Zitterbewegung' in General Relativity ☆

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ABSTRACT

We show how the Dixon's system of first order equations of motion for the particle with inner dipole structure together with the side Mathisson constraint follows from rather general construction of the 'Hamilton system' developed by Weyssenhoff, Rund and Grässer to describe the phase space counterpart of the evolution under the ordinary Euler–Poisson differential equation of the parameter-invariant variational problem with second derivatives. One concrete expression of the 'Hamilton function' leads to the General Relativistic form of the fourth order equation of motion known to describe the quasi-classical 'quiver' particle in Special Relativity. The corresponding Lagrange function including velocity and acceleration coincides in the flat space of Special Relativity with the one considered by Bopp in an attempt to give an approximate variational formulation of the motion of self-radiating electron, when expressed in terms of geometric quantities.

1. Introduction

Consider a quite popular and fairly general Dixon [1] system of first order ordinary differential equations[1]

$$\begin{cases} P'_\alpha = -\dfrac{1}{2} R_{\alpha\beta}{}^{\rho\nu} \dot{x}^\beta S_{\rho\nu}, \\ S'_{\alpha\beta} = P_\alpha \dot{x}_\beta - P_\beta \dot{x}_\alpha, \end{cases} \qquad S_{\alpha\beta} + S_{\beta\alpha} = 0, \tag{1}$$

written in terms of the covariant derivatives, denoted from here on by *prime*.

In the theory of General Relativity such equations should hold along the world line of a quasi-classical particle endowed with the inner angular momentum (said 'spin') $S^{\alpha\beta}$, responsible for its dipole structure.

Among several additional side conditions needed to make system (1) solvable (see [2]), we choose to focus on the one preferred by Mathisson [3]

$$\dot{x}^\rho S_{\rho\alpha} = 0. \tag{2}$$

Now imagine that someone wishes to construct a sort of 'Hamilton' picture of the system (1) under the imposed constraint (2). There exists a non-conventional approach to do this along the following guidelines. First, try to eliminate the variables $S^{\alpha\beta}$ by means of taking subsequent differential prolongations of (1), (2). Further, try to find a variational problem

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[1] Our definition of the curvature tensor differs in sign from the one adopted in papers [1,2].

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with higher derivatives for thus obtained equations, perhaps, under different constraints. Then pass to the corresponding Hamilton–Ostrohrads'kyj counterpart in terms of the generalized momenta. As the last step, compose some geometric quantities $S^{\alpha\beta}$ from the canonical variables: the momenta and the velocities. If successful, one regains the system (1), with the constraint (2) already satisfied identically.

We show to the end of this paper that (1) *follows from a fairly general setting of the second order parameter-invariant variational problem as its 'Hamiltonian' counterpart by the appropriate definition of* $S^{\alpha\beta}$.

In flat space–time of Special Relativity the differential elimination of the variable $S^{\alpha\beta}$ from (1), (2) leads to the fourth order equation of motion

$$\ddddot{x} + \left(k^2 - \frac{m^2}{\sigma^2}\right)\ddot{x} = 0, \qquad (\dot{x}\cdot\dot{x}) = 1, \tag{3}$$

where $k^2 = (\ddot{x}\cdot\ddot{x})$ is the first integral of (3), and

$$\sigma_\alpha = \frac{1}{2\|u\|}\epsilon_{\alpha\beta\rho\nu}u^\beta S^{\rho\nu}. \tag{4}$$

Eq. (3) was shown by Riewe [4] and Costantelos [5] to describe 'Zitterbewegung' (quiver) of a quasi-classical particle.

We show to the end of this paper that (3) *occurs as the natural parameterization of an Euler–Poisson* (*said variational*) *equation constrained to the manifold* $k = k_0$, *of some parameter-invariant variational problem of the second order.*

The above program for the flat space–time was carried out in two preceding papers [6,7].[2]

In present paper rather that go all way round the procedure mentioned above, we merely offer a straightforward generalization of the 'Hamiltonian' depiction obtained in [7] to the case of (pseudo)Riemannian geometry.

2. The Grässer–Rund–Weyssenhoff canonical equations

In the space of the fourth order Ehresmann velocities T^4M let us stick to the commonly recognized coordinates $x = \{x^\alpha\} \in M$, $u = \dot{x} = \frac{dx}{d\tau}(0)$, $\dot{u} = \frac{d^2x}{d\tau^2}(0)$, $\ddot{u} = \frac{d^3x}{d\tau^3}(0)$, $\dddot{u} = \frac{d^4x}{d\tau^4}(0)$. A function $\mathcal{L}(x, u, \dot{u})$ defined on T^2M, constitutes a parameter-invariant variational problem $\delta\int\mathcal{L}\,d\tau = 0$ if and only if it satisfies the now well-known Zermelo conditions:

$$u^\alpha\frac{\partial\mathcal{L}}{\partial\dot{u}^\alpha} \equiv 0, \tag{5.1}$$

$$u^\alpha\frac{\partial\mathcal{L}}{\partial u^\alpha} + 2\dot{u}^\alpha\frac{\partial\mathcal{L}}{\partial\dot{u}^\alpha} - \mathcal{L} \equiv 0. \tag{5.2}$$

We also recall the definition of the Legendre transformation, that is the mapping $Le : T^3M \to T^*(TM)$ over TM given by

$$\wp^{(1)} = \frac{\partial\mathcal{L}}{\partial\dot{u}}, \tag{6.1}$$

$$\wp = \frac{\partial\mathcal{L}}{\partial u} - \mathcal{D}_\tau\wp^{(1)}, \tag{6.2}$$

where

$$\mathcal{D}_\tau = u\frac{\partial}{\partial x} + \dot{u}\frac{\partial}{\partial u} + \ddot{u}\frac{\partial}{\partial\dot{u}}$$

denotes the operator of total derivative, and the canonical coordinates in $T^*(TM)$ are denoted by x, u, p, $p^{(1)}$. Applying $\mathcal{D}_\tau$ to (5.1) immediately gives that in terms of the mixed set of variables $\{\dot{u}, p, p^{(1)}\}$ the Zermelo conditions look like

$$Z_1 \overset{\text{def}}{=} u^\alpha\wp^{(1)}{}_\alpha = 0, \tag{7.1}$$

$$Z_2 \overset{\text{def}}{=} u^\alpha\wp_\alpha + \dot{u}^\alpha\wp^{(1)}{}_\alpha - \mathcal{L} = 0. \tag{7.2}$$

The standard Liouville form Λ on $T^*(TM)$ reads

$$\Lambda = p.dx + p^{(1)}.du.$$

The system of the canonical equations developed in the paper of Grässer [8], who took as a basis the works of Rund [9] and Weyssenhoff [10], follow from the exterior differential equation

$$Le^{-1}i_X\,d\Lambda = -\lambda Le^{-1}\,d\mathcal{H} - \mu Le^{-1}\,dZ_1. \tag{8}$$

[2] A technical mistake that slipped in the expression for the Hamilton function in paper [6] has been corrected in paper [7].