



A representation-theoretical proof of Branson's classification of elliptic generalized gradients

Mihaela Pilca ^{a,b,*}

^a NWF I, Universität Regensburg, Universitätsstr. 31, D-93040 Regensburg, Germany

^b Institute of Mathematics "Simion Stoilow" of the Romanian Academy, 21 Calea Grivitei Str., 010702 Bucharest, Romania

ARTICLE INFO

Article history:

Available online 19 April 2011

Communicated by M.G. Eastwood

MSC:

58J10

22E45

Keywords:

Generalized gradient

Elliptic differential operator

Kato constant

Kato inequality

ABSTRACT

The purpose of this paper is to present a new proof of Branson's classification (Branson, 1997 [3]), of minimal elliptic sums of generalized gradients. The advantage of this proof is that it is local, being mainly based on representation theory and on the relationship between ellipticity and refined Kato inequalities. This approach is promising for the classification of elliptic generalized gradients of G -structures, for other subgroups G of the special orthogonal group.

© 2011 Elsevier B.V. All rights reserved.

1. Introduction

Generalized gradients, also called Stein–Weiss operators, were first introduced by E. Stein and G. Weiss, [16], as a generalization of the Cauchy–Riemann equations. They are first order differential operators acting on sections of vector bundles associated to irreducible representations of the special orthogonal group (or of the spin group if the manifold is spin), which are given by the projections of a metric covariant derivative onto irreducible subbundles.

On an oriented Riemannian manifold, generalized gradients naturally give rise, by composition with their formal adjoints, to second order differential operators acting on sections of associated vector bundles. Particularly important are linear combinations of such second order operators which are elliptic. These were classified by Th. Branson, [3], who showed that it is enough to take surprisingly few generalized gradients in order to obtain an elliptic operator. It thus turned out that Laplace-type operators represent the generic case. The argument of Th. Branson relies on tools and techniques of harmonic analysis and explicit computations of the spectra of generalized gradients on the sphere. Partial results regarding the ellipticity of natural first order operators were previously obtained by J. Kalina, A. Pierzchalski and P. Walczak, [10], who showed that the only generalized gradient which is strongly elliptic is given by the projection onto the Cartan summand. Furthermore, the projection onto its complement is also elliptic, by a result of E. Stein and G. Weiss, [16].

In this paper we present a local proof of Branson's classification, up to one exceptional case. We shall only sketch the main ideas, which are developed in [14] and will appear elsewhere. Our method is completely different from the original one in [3], which seems to be specific for the two structure groups $SO(n)$ or $Spin(n)$. The starting point is the remark that these elliptic operators are closely related to the existence of refined Kato inequalities, which was first noticed by Bourguignon, [1]. In the first part of our proof we extend to all generalized gradients the explicit computation of the

* Correspondence to: NWF I, Universität Regensburg, Universitätsstr. 31, D-93040 Regensburg, Germany.

E-mail address: Mihaela.Pilca@mathematik.uni-regensburg.de.

optimal Kato constants for the elliptic ones, which was given by Calderbank, Gauduchon and Herzlich, [7]. We notice that a different approach to establishing a general formula for the optimal Kato constants has been given by Branson, [5]. In the second part of the proof we show that these are *all* minimal elliptic operators, using the branching rule for the special orthogonal group.

2. Branson's classification of elliptic generalized gradients

In this section we recall Branson's classification of elliptic generalized gradients, cf. [3].

Let (M, g) be an oriented Riemannian manifold, $SO_g M$ denotes the principal $SO(n)$ -bundle of oriented orthonormal frames and ∇ any metric connection. If M has, in addition, a spin structure, then we consider the corresponding principal $Spin(n)$ -bundle, $Spin_g M$, and the induced metric connection ∇ . We consider vector bundles $V_\lambda M$, associated to $SO_g M$ (or $Spin_g M$) and irreducible $SO(n)$ - (or $Spin(n)$)-representations of highest weight λ , with the induced connection ∇ .

With a slight abuse of notation, we use the same symbol for an irreducible representation and its highest weight. We denote by τ the (complex) standard representation and the coordinates $(\lambda_1, \dots, \lambda_m)$ of a weight λ are given with respect to the basis $\{\varepsilon_1, \dots, \varepsilon_m\}$ of $\mathfrak{h}^*$, which is the dual of the basis $\{e_1 \wedge e_2, \dots, e_{2m-1} \wedge e_{2m}\}$ that fixes a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{so}(n)$, where $n = 2m$ or $n = 2m + 1$ and $\{e_1, \dots, e_n\}$ is an oriented orthonormal basis of $\mathbb{R}^n$. The so-called *classical selection rule* (see [8]) holds: an irreducible representation of highest weight μ occurs in the decomposition of $\tau \otimes \lambda$ if and only if the following two conditions are fulfilled

- (i) $\mu = \lambda \pm \varepsilon_j$, for some $j = 1, \dots, m$, or $n = 2m + 1$, $\lambda_m > 0$ and $\mu = \lambda$,
- (ii) μ is a dominant weight, i.e. $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{Z}^m \cup (\frac{1}{2} + \mathbb{Z})^m$ and

$$\mu_1 \geq \mu_2 \geq \dots \geq \mu_{m-1} \geq |\mu_m|, \quad \text{if } n = 2m, \quad \text{or} \quad \mu_1 \geq \mu_2 \geq \dots \geq \mu_m \geq 0, \quad \text{if } n = 2m + 1. \quad (2.1)$$

We write $\varepsilon \subset \lambda$ for the weights ε of τ such that $\lambda + \varepsilon$ occurs in the decomposition of $\tau \otimes \lambda$ and call them *relevant* for λ . This decomposition is multiplicity-free, i.e. the isotypical components are actually irreducible, so that the projections Π_ε onto each irreducible summand $\lambda + \varepsilon$ are well-defined. The decomposition carries over to the associated vector bundles:

$$T^*M \otimes V_\lambda M \cong TM \otimes V_\lambda M \cong \bigoplus_{\varepsilon \subset \lambda} V_{\lambda+\varepsilon} M, \quad (2.2)$$

where the corresponding projections are also denoted by Π_ε .

Definition 2.1. For each relevant weight ε of λ , i.e. for each irreducible component in the decomposition of $T^*M \otimes V_\lambda M$, there is a *generalized gradient* P_ε defined by the composition:

$$\Gamma(V_\lambda M) \xrightarrow{\nabla} \Gamma(T^*M \otimes V_\lambda M) \xrightarrow{\Pi_\varepsilon} \Gamma(V_{\lambda+\varepsilon} M). \quad (2.3)$$

Generalized gradients may be thus defined by any metric connection. Those defined by the Levi-Civita connection play an important role since they are conformal invariant [13]. Some of the most important first order differential operators which naturally appear in geometry are, up to normalization, particular cases of generalized gradients. For example, on a Riemannian manifold, the exterior differential acting on differential forms, its formal adjoint, the codifferential, and the conformal Killing operator on 1-forms are generalized gradients. On a spin manifold, classical examples of generalized gradients are the Dirac operator, the twistor (or Penrose) operator and the Rarita–Schwinger operator.

Remark 2.2. Essentially the same construction as above may be used to define generalized gradients associated to a G -structure. For a study of these G -generalized gradients, where G is one of the subgroups of $SO(n)$ from Berger's list of holonomy groups, we refer the reader e.g. to [12].

Since the principal symbol of a generalized gradient P_ε is given by the projection Π_ε defining it, it follows that P_ε is overdetermined (or injectively) elliptic (in the sequel we shall shortly say elliptic) if and only if the map $\Pi_\varepsilon \circ (\xi \otimes \cdot) : V_\lambda \rightarrow V_{\lambda+\varepsilon}$ is injective, for each nonzero section $\xi \in \Gamma(T^*_\lambda M)$. Thus, the generalized gradient P_ε is (strongly) injectively elliptic if and only if Π_ε is not vanishing on each nonzero decomposable element.

For any subset I of the set of relevant weights of λ , we consider the following second order differential operator: $\sum_{\varepsilon \in I} P_\varepsilon^* P_\varepsilon$, where $P_\varepsilon := \Pi_\varepsilon \circ \nabla$ is the generalized gradient. In [3], Branson completely classified the operators of this type which are elliptic.

The problem may be reduced to first order differential operators as follows: if $P_I := \sum_{\varepsilon \in I} P_\varepsilon$, then $\sum_{\varepsilon \in I} P_\varepsilon^* P_\varepsilon$ is elliptic if and only if P_I is elliptic, i.e. the projection $\Pi_I := \sum_{\varepsilon \in I} \Pi_\varepsilon : T \otimes V_\lambda \rightarrow \bigoplus_{\varepsilon \in I} V_{\lambda+\varepsilon}$ is injective when restricted to the set of decomposable elements in $T \otimes V_\lambda$. Thus, the study of the ellipticity is reduced to a question on the representation theory of $\mathfrak{so}(n)$, without reference to any particular manifold.

Download English Version:

<https://daneshyari.com/en/article/4606364>

Download Persian Version:

<https://daneshyari.com/article/4606364>

[Daneshyari.com](https://daneshyari.com)