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Hamiltonian structure of the Yang–Mills functional

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ABSTRACT

Hamilton equations based upon a general Lepagean equivalent of the Yang–Mills Lagrangian are investigated. A regularization of the Yang–Mills Lagrangian which is singular with respect to the standard regularity conditions is derived.

1. Introduction

The aim of this paper is to study a Hamiltonian structure of the Yang–Mills theory. Many authors have worked on a geometric formulation of Hamiltonian theory (cf. [2,3,8,9] and references therein). The approach of the calculus of variations on fibered manifolds which is based on Krupková's concept of a Lepagean $(n+1)$-form generalizing Krupka's concept of a Lepagean n-form is adopted (n is the dimension of the base manifold of the fibered manifold). This approach opens a possibility to regularize a Lagrangian whose standard Hessian is singular, it is the case of the Yang–Mills Lagrangian too.

This informative paper is organized as follows. In Section 2, a survey of the general variational theory [1,5,6] is given and the results involving a Hamiltonian field theory [10,11] needed in the Yang–Mills theory are summarized. In Section 3, a Hamiltonian system associated with the Yang–Mills theory is presented and Legendre transformation after the regularization is performed.

2. Lagrangian and Hamiltonian theory on fibered manifolds

Recall our standard notation [5–9]. We have a fibered manifold $\pi : Y \to X$, and write $n = \dim X$, $n + m = \dim Y$. J^rY is the *r-jet prolongation* of Y, and $\pi^{r,s} : J^rY \to J^sY$, $\pi^r : J^rY \to X$ are the *canonical jet projections*. The *r-jet prolongation* of a section γ is defined to be the mapping $x \to J^r\gamma(x) = J^r_x\gamma$. For any set $W \subset Y$ we denote $W^r = (\pi^{r,0})^{-1}(W)$. Any *fibered chart* (V, ψ), $\psi = (x^i, y^\sigma)$, on Y induces the *associated charts* on X and J^rY, denoted by (U, φ), $\varphi = (x^i)$, and (V^r, ψ^r), $\psi^r = (x^i, y^\sigma, y^\sigma_{j_1}, y^\sigma_{j_1 j_2}, \ldots, y^\sigma_{j_1 j_2 \ldots j_r})$, respectively; here $1 \leqslant i, j_1, \ldots, j_r \leqslant n$, $1 \leqslant \sigma \leqslant m$, and $V^r = (\pi^{r,0})^{-1}(V)$, $U = \pi^r(V)$.

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We denote $\omega_0 = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$, and its contractions $\omega_i = i_{\partial/\partial x^i}\omega_0$, $\omega_{ij} = i_{\partial/\partial x^j}\omega_i$. We define the *formal derivative operator* with respect to x^i by $d_i = \partial/\partial x^i + y^\sigma_i(\partial/\partial y^\sigma) + y^\sigma_{j_1 i}(\partial/\partial y^\sigma_{j_1}) + \cdots + y^\sigma_{j_1 j_2 \ldots j_r i}(\partial/\partial y^\sigma_{j_1 j_2 \ldots j_r})$.

For any open set $W \subset Y$, let $\Omega^r_0 W$ be the ring of functions on W^r. The $\Omega^r_0 W$-module of differential q-forms on W^r is denoted by $\Omega^r_q W$, and the exterior algebra of forms on W^r is denoted by $\Omega^r W$. The module of π^r-horizontal q-forms is denoted by $\Omega^r_{q,X} W$. A form $\rho \in \Omega^r_q W$ is called *$\pi^{r,s}$-projectable* if there exists a form $\rho_0 \in \Omega^s_q W$ such that $\pi^{r,s*}\rho_0 = \rho$. The fibered structure of Y induces a morphism of exterior algebras $h : \Omega^r W \to \Omega^{r+1} W$, called the *horizontalization*. In a fibered chart h is defined by $hf = f \circ \pi^{r+1,r}$, $h\,dx^i = dx^i$, $h\,dy^\sigma_{j_1 j_2 \ldots j_p} = y^\sigma_{j_1 j_2 \ldots j_p i}\,dx^i$, where f is a real function on W^r, and $0 \leqslant p \leqslant r$.

We say that a form $\eta \in \Omega^r_q W$ is contact, if $h\eta = 0$. For any fibered chart the 1-forms $\eta^\sigma_{j_1 j_2 \ldots j_p} = dy^\sigma_{j_1 j_2 \ldots j_p} - y^\sigma_{j_1 j_2 \ldots j_p i}\,dx^i$, where $1 \leqslant p \leqslant r-1$, are examples of contact 1-forms, defined on V^r. Note that these forms define a basis of 1-forms on V^r, $(dx^i, \eta^\sigma_{j_1 j_2 \ldots j_p}, dy^\sigma_{j_1 j_2 \ldots j_r})$. For every $\eta \in \Omega^r_q W$ we have a unique canonical decomposition $\pi^{r+1,r*}\eta = h\eta + p_1\eta + \cdots + p_q\eta$ into a sum of a horizontal form $h\eta$ and *k-contact* forms $p_k\eta$, $1 \leqslant k \leqslant q$; in coordinates each term of a k-contact form with respect to a basis $(dx^i, \eta^\sigma_{j_1 j_2 \ldots j_p}, dy^\sigma_{j_1 j_2 \ldots j_{r+1}})$, $1 \leqslant p \leqslant r$, contains exactly k of the 1-contact 1-forms $\eta^\sigma_{j_1 j_2 \ldots j_p}$.

A *Lagrangian* for Y is a π^r-horizontal n-form λ on the r-jet prolongation $J^r Y$ of Y. The number r is called the *order* of λ. In a fibered chart (V, ψ) on Y, and the associated chart on $J^r Y$, a Lagrangian of order r has an expression $\lambda = \mathcal{L}\omega_0$, where $\mathcal{L} : V^r \to \mathbb{R}$ is the *component* of λ with respect to (V, ψ) (the *Lagrange function* associated with (V, ψ)). The *Euler–Lagrange form* of λ is defined to be an $(n+1)$-form E_λ on $J^{2r} Y$, defined by

$$E_\lambda = E_\sigma(\mathcal{L})\eta^\sigma \wedge \omega_0, \quad E_\sigma(\mathcal{L}) = \sum_{l=0}^{r} (-1)^l d_{j_1} d_{j_2} \ldots d_{j_l} \frac{\partial \mathcal{L}}{\partial y^\sigma_{j_1 j_2 \ldots j_l}},$$

$E_\sigma(\mathcal{L})$ are the *Euler–Lagrange expressions*.

An n-form ρ on $J^s Y$ is called a *Lepagean n-form* (*of order s*) if the $(n+1)$-form $p_1\,d\rho$ is $\pi^{s+1,0}$-horizontal. If $h\rho = \lambda$ then we say that ρ is a *Lepagean equivalent* of the Lagrangian λ. An $(n+1)$-form E on $J^s Y$, $s \geqslant 1$, is called a *dynamical form* if it is 1-contact and $\pi^{s,0}$-horizontal, i.e. in any fibered chart $E = E_\sigma \eta^\sigma \wedge \omega_0$, where E_σ are functions on an open set in $J^s Y$. A closed $(n+1)$-form α on $J^s Y$, $s \geqslant 0$, is called a *Lepagean $(n+1)$-form* if $p_1\alpha$ is a dynamical form. If α is a Lepagean $(n+1)$-form and $p_1\alpha = E$ then we say that α is a *Lepagean equivalent* of E.

We say that Lepagean $(n+1)$-forms α_1 and α_2 are *equivalent* if (up to a projection) $p_1\alpha_1 = p_1\alpha_2$. The equivalence class $[\alpha]$ of all Lepagean $(n+1)$-forms is called a *Lagrangian system*. Let $s \geqslant 0$ denote the *dynamical order* of the Lagrangian system $[\alpha]$ defined as the minimum of the set of orders of the forms from $[\alpha]$, then a section $\gamma : U \to Y$ defined on an open subset $U \subset X$ is an *extremal* of $E = p_1\alpha$, i.e. $E \circ J^{s+1}\gamma = 0$, iff for every π-vertical vector field ξ on Y,

$$J^s\gamma^* i_{J^s\xi}\alpha = 0, \tag{1}$$

where α is any representative of order s of $[\alpha]$. Eqs. (1) are called *Euler–Lagrange equations* corresponding to the Lagrangian system $[\alpha]$.

A *Hamiltonian system of order s* is given by a Lepagean $(n+1)$-form α on $J^s Y$. A section δ of the fibered manifold π^s is called a *Hamilton extremal of α* if

$$\delta^* i_\xi \alpha = 0, \tag{2}$$

for every π^s-vertical vector field ξ on $J^s Y$. Eqs. (2) are called *Hamilton equations* of α. If there exists an at most k-contact Lepagean n-form ρ, $1 \leqslant k \leqslant n$, in a neighborhood of every point in $J^s Y$ such that $\alpha = d\rho$, we call Eqs. (2) *Hamilton p_k-equations*.

For a Lagrangian $\lambda = \mathcal{L}\omega_0 \in \Omega^1_{n,X} W$ which is singular in the standard sense, i.e. the regularity condition $\det(\frac{\partial^2 \mathcal{L}}{\partial y^\sigma_i \partial y^\nu_j}) \neq 0$ at each point of W^1 is not satisfied, we can consider its simple *regularization* $\rho \in \Omega^1_n W$, defined as a Lepagean equivalent of λ, such that ρ is at most 2-contact, $p_2\rho = p_2\beta$ for a $\pi^{1,0}$-projectable form $\beta \in \Omega^1_n W$, i.e. using a fibered chart

$$\rho = \mathcal{L}\omega_0 + \frac{\partial \mathcal{L}}{\partial y^\sigma_i}\eta^\sigma \wedge \omega_i + g^{ij}_{\sigma\nu}\eta^\sigma \wedge \eta^\nu \wedge \omega_{ij}, \tag{3}$$

where $g^{ij}_{\sigma\nu} = -g^{ji}_{\sigma\nu} = -g^{ij}_{\nu\sigma}$ are functions on W and ρ satisfies the following regularity condition on W^1

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial y^\sigma_i \partial y^\nu_j} - 4g^{ij}_{\sigma\nu}\right) \neq 0,$$

where (σ, i) labels rows and (ν, j) labels columns. The next statements were proved in [10,11].

Theorem 2.1. *Let ρ be the regularization of λ as above (3) and $\alpha = d\rho$.*

(1) *Then every Hamilton extremal δ of $\alpha = d\rho$ is of the form $\delta = J^1\gamma$, where γ is an extremal of $E_\lambda = p_1\,d\rho$.*