Differential Geometry and its Applications 29 (2011) S255–S260

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Curvatures of tangent hyperquadric bundles

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ARTICLE INFO

Article history:
Available online 2 May 2011
Communicated by O. Kowalski

MSC:
53C07
53C21

Keywords:
Riemannian manifold
Tangent bundle
Tangent hyperquadric bundle
Sasaki metric
Ricci curvature
Scalar curvature

ABSTRACT

We study geometry of tangent hyperquadric bundles over pseudo-Riemannian manifolds, which are equipped, as submanifolds of the tangent bundles, with the induced Sasaki metric. All kinds of curvatures are calculated, and geometric results concerning the Ricci curvature and the scalar curvature are proved. There exists a hyperquadric bundle whose scalar curvature is a preassigned constant.

0. Introduction

Geometry of tangent sphere bundles over a Riemannian manifold has been studied by many authors. We refer to Boeckx and Vanhecke [1] and Calvaruso [2] for survey on geometry of unit tangent sphere bundles. O. Kowalski and the second author have studied in [4] geometry of the tangent sphere bundle of constant radius with the metric induced by the Sasaki metric on the tangent bundle over a Riemannian manifold. They have shown that geometry of such bundles depends not only on geometry of the base manifold but also on the radius (see also survey [6] for their extensive research on this topic).

In this paper we assume that the base manifold (M, g) is a pseudo-Riemannian manifold and consider the bundle $T_r^\epsilon M$ called the tangent hyperquadric bundle of radius r, which corresponds to the tangent sphere bundle over a Riemannian manifold. We induce the metric $\tilde{g}$ on $T_r^\epsilon M$ by the Sasaki metric on the tangent bundle TM over (M, g).

We shall define in Section 1 the tangent hyperquadric bundle $T_r^\epsilon M$ and derive formulas on the Levi-Civita connection of $\tilde{g}$. Then, we shall give in Section 2 basic formulas on the curvatures of $\tilde{g}$. We shall study properties of the scalar curvature in Section 3 which generalize some results in [4] and [5]. Yet, some of our results in this paper are proper to the bundles over pseudo-Riemannian manifolds with indefinite metrics. In particular, we shall prove that there exists a tangent hyperquadric bundle whose scalar curvature is a preassigned constant.

The tangent hyperquadric bundles have been studied by Dragomir and Perrone [3] on different topics from ours.

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0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2011.04.050

1. Tangent hyperquadric bundles

Let M be a smooth and connected n-manifold. Then the tangent bundle TM over M consists of all pairs (x,u), where x is a point of M and u is a vector from the tangent space M_x of M at x. We denote by p the natural projection of TM to M defined by $p(x,u)=x$.

Let (M,g) be a pseudo-Riemannian manifold and ∇ its Levi-Civita connection. Then the tangent space $(TM)_{(x,u)}$ of TM at (x,u) splits into the horizontal and vertical subspaces $H_{(x,u)}$ and $V_{(x,u)}$ with respect to ∇:

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

For a vector $X \in M_x$, the *horizontal lift* of X to a point $(x,u) \in TM$ is a unique vector $X^h \in H_{(x,u)}$ such that $p_* X^h = X$. The *vertical lift* of X to (x,u) is a unique vector $X^v \in V_{(x,u)}$ such that $X^v(\mathrm{d}f) = Xf$ for all smooth functions f on M. Here we consider a one-form $\mathrm{d}f$ on M as a function on TM defined by $(\mathrm{d}f)(x,u) = uf$ for all $(x,u) \in TM$. The map $X \mapsto X^h$ is an isomorphism between M_x and $H_{(x,u)}$; and the map $X \mapsto X^v$ is an isomorphism between M_x and $V_{(x,u)}$. In an obvious way we can define the horizontal and vertical lifts of vector fields on M. These are uniquely defined vector fields on TM.

To each system of local coordinates $(x^1, x^2, \dots, x^n)$ in M, one defines, in the standard way, the system of local coordinates $(x^1, x^2, \dots, x^n; u^1, u^2, \dots, u^n)$ in TM. The *canonical vertical vector field* on TM is a vector field $\boldsymbol{U}$ defined, in terms of local coordinates, by $\boldsymbol{U} = \sum_i u^i \partial/\partial u^i$. Here $\boldsymbol{U}$ does not depend on the choice of local coordinates and it is defined globally on TM. For a vector $u = \sum_i u^i (\partial/\partial x^i)_x \in M_x$, we see that $u^h_{(x,u)} = \sum_i u^i (\partial/\partial x^i)^h_{(x,u)}$ and $u^v_{(x,u)} = \sum_i u^i (\partial/\partial x^i)^v_{(x,u)} = \boldsymbol{U}_{(x,u)}$.

The *Sasaki metric* on the tangent bundle TM of a pseudo-Riemannian manifold (M,g) is determined, at each point $(x,u) \in TM$, by the formulas

$$\begin{aligned} \bar{g}_{(x,u)}\big(X^h, Y^h\big) &= g_x(X,Y), \\ \bar{g}_{(x,u)}\big(X^h, Y^v\big) &= 0, \\ \bar{g}_{(x,u)}\big(X^v, Y^v\big) &= g_x(X,Y), \end{aligned} \tag{1.1}$$

where X and Y are arbitrary vectors from M_x. Let $\bar{\nabla}$ be the Levi-Civita connection of $(TM, \bar{g})$, and let X and Y be vector fields on M. Then, at each *fixed* point $(x,u) \in TM$,

$$\begin{aligned} \big(\bar{\nabla}_{X^h} Y^h\big)_{(x,u)} &= (\nabla_X Y)^h_{(x,u)} - \frac{1}{2}\big(R_x(X,Y)u\big)^v, \\ \big(\bar{\nabla}_{X^h} Y^v\big)_{(x,u)} &= \frac{1}{2}\big(R_x(u,Y)X\big)^h + (\nabla_X Y)^v_{(x,u)}, \\ \big(\bar{\nabla}_{X^v} Y^h\big)_{(x,u)} &= \frac{1}{2}\big(R_x(u,X)Y\big)^h, \\ \big(\bar{\nabla}_{X^v} Y^v\big)_{(x,u)} &= 0, \end{aligned} \tag{1.2}$$

where R is the Riemannian curvature tensor of (M,g) defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. As concerns the canonical vertical vector field $\boldsymbol{U}$, we have

$$\begin{aligned} &\bar{\nabla}_{X^h} \boldsymbol{U} = 0, \qquad \bar{\nabla}_{X^v} \boldsymbol{U} = X^v, \\ &\bar{\nabla}_{\boldsymbol{U}} X^h = 0, \qquad \bar{\nabla}_{\boldsymbol{U}} X^v = 0, \\ &\bar{\nabla}_{\boldsymbol{U}} \boldsymbol{U} = \boldsymbol{U} \end{aligned} \tag{1.3}$$

for each vector field X on M.

Let r be a positive number. We set $\epsilon = 1$ if g is positive definite, $\epsilon = -1$ if g is negative definite, and $\epsilon = \pm 1$ if g is indefinite. The *tangent hyperquadric bundle of radius* r over a pseudo-Riemannian manifold (M,g) is a hypersurface $T^\epsilon_r M = \{(x,u) \in TM \mid g_x(u,u) = \epsilon r^2\}$ of TM. If g is positive definite, then $T^\epsilon_r M = T_r M$ is the tangent sphere bundle with radius r (see for example [4]).

The canonical vertical vector field $\boldsymbol{U}$ is normal to $T^\epsilon_r M$ in $(TM, \bar{g})$ at each point $(x,u) \in T^\epsilon_r M$. Also, $\bar{g}(\boldsymbol{U}, \boldsymbol{U}) = \epsilon r^2$ along $T^\epsilon_r M$. For any vector field X tangent to M, the horizontal lift X^h is always tangent to $T^\epsilon_r M$ at each point $(x,u) \in T^\epsilon_r M$. Yet, in general, the vertical lift X^v is not tangent to $T^\epsilon_r M$ at (x,u). The *tangential lift* X^t of X is a vector field on $T^\epsilon_r M$ defined by

$$X^t = X^v - \epsilon \frac{1}{r^2} \bar{g}\big(X^v, \boldsymbol{U}\big)\boldsymbol{U}.$$

Thus at each point $(x,u) \in T^\epsilon_r M$, we have

$$X^t_{(x,u)} = X^v_{(x,u)} - \epsilon \frac{1}{r^2} g_x(X,u) \boldsymbol{U}_{(x,u)}.$$