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Heat kernel for open manifolds

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ABSTRACT

It is known that for open manifolds with bounded geometry, the differential form heat kernel exists and is unique. Furthermore, it has been shown that the components of the differential form heat kernel are related via the exterior derivative and the coderivative. We will give a proof of this condition for complete manifolds with Ricci curvature bounded below, and then use it to give an integral representation of the heat kernel of degree k.

1. Introduction

In this paper we are considering the differential forms heat equation on manifolds, in particular we are considering $(\Delta + \partial_t)\omega = 0$ with Dirichlet initial conditions. Our goal is to produce a formula for the Green's function, also known as the heat kernel or fundamental solution, which gives the solution of this equation.

The solutions of this equation in the case of functions, or 0-forms, is well-known. The work on differential forms has been much more recent. In 1983, Dodziuk [4] proved that for complete oriented C^∞ Riemannian manifolds with Ricci curvature bounded below, bounded solutions are uniquely determined by their initial values. In a 1988 paper by Buttig, [1], the author listed in Appendix A.2 properties of a "good heat kernel". In 1991, Buttig and Eichhorn [2] were able to give an existence and uniqueness proof for Buttig's conjecture for the differential forms heat kernel on open manifolds of bounded geometry. One of the properties given by Buttig and Eichhorn for a global heat kernel was that the heat kernels $K_k(\mathbf{x}, \mathbf{y}, t)$ and $K_{k+1}(\mathbf{x}, \mathbf{y}, t)$ are related by

$$d_{\mathbf{x}} K_k(\mathbf{x}, \mathbf{y}, t) = d^*_{\mathbf{y}} K_{k+1}(\mathbf{x}, \mathbf{y}, t). \tag{1.1}$$

Here, K_k refers to the degree k portion of the heat kernel. We will use the terminology "k-form heat kernel" to refer to the degree k component of the heat kernel. Using that identity (1.1), we have previously shown, [5], that the 1-form heat kernel on open Riemann surfaces of bounded geometry has the form

$$K_1(\mathbf{x}, \mathbf{y}, t) = (I + *_{\mathbf{x}} *_{\mathbf{y}})\, d_{\mathbf{x}}\, d_{\mathbf{y}} \int_t^\infty K_0(\mathbf{x}, \mathbf{y}, \tau)\, d\tau.$$

This directly relates the 1-form heat kernel to the 0-form heat kernel, about which more is known.

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In this article, we will present a proof of this property for manifolds with Ricci curvature bounded below, and then use this to give a formula for the k-form heat kernel.

2. Ricci curvature bounded below

To start, we will established the identity (1.1) for complete manifolds with Ricci curvature bounded below. We use the condition on the Ricci curvature to guarantee the existence and uniqueness of the differential forms heat kernel.

Lemma 2.1. *For a complete manifold M, with Ricci curvature bounded from below, we have the following relationship between the k- and $(k+1)$-form heat kernels*:

1. $d_{\mathbf{x}} K_k(\mathbf{x},\mathbf{y},t) = d^*_{\mathbf{y}} K_{k+1}(\mathbf{x},\mathbf{y},t)$,
2. $d_{\mathbf{y}} K_k(\mathbf{x},\mathbf{y},t) = d^*_{\mathbf{x}} K_{k+1}(\mathbf{x},\mathbf{y},t)$.

Proof. Let $E(\mathbf{x},\mathbf{y},t) = d_{\mathbf{x}} K_k(\mathbf{x},\mathbf{y},t) - d^*_{\mathbf{y}} K_{k+1}(\mathbf{x},\mathbf{y},t)$. We will demonstrate that E satisfies the heat equation with zero as the initial condition. This will imply, by uniqueness of the solutions of the heat equation, see [4], that $E \equiv 0$, giving the desired result.

First

$$\begin{aligned}\Delta_{\mathbf{x}} E &= \Delta_{\mathbf{x}}\, d_{\mathbf{x}} K_k(\mathbf{x},\mathbf{y},t) - \Delta_{\mathbf{x}}\, d^*_{\mathbf{y}} K_{k+1}(\mathbf{x},\mathbf{y},t)\\ &= d_{\mathbf{x}} \Delta_{\mathbf{x}} K_k(\mathbf{x},\mathbf{y},t) - d^*_{\mathbf{y}} \Delta_{\mathbf{x}} K_{k+1}(\mathbf{x},\mathbf{y},t)\\ &= d_{\mathbf{x}}(-\partial_t) K_k(\mathbf{x},\mathbf{y},t) - d^*_{\mathbf{y}}(-\partial_t) K_{k+1}(\mathbf{x},\mathbf{y},t)\\ &= -\partial_t E.\end{aligned}$$

Next consider $W := \langle E, \omega(\mathbf{x})\rangle$, where ω is a suitable test function and $\langle \mu,\nu\rangle = \int_M \mu \wedge *\nu$. Then

$$\begin{aligned}\lim_{t\to 0+} W &= \lim_{t\to 0+} \big\langle d_{\mathbf{x}} K_k, \omega(\mathbf{x})\big\rangle - \big\langle d^*_{\mathbf{y}} K_{k+1}, \omega(\mathbf{x})\big\rangle\\ &= \lim_{t\to 0+} \big\langle K_k, d^*_{\mathbf{x}}\omega(\mathbf{x})\big\rangle - d^*_{\mathbf{y}}\big\langle K_{k+1}, \omega(\mathbf{x})\big\rangle\\ &= d^*_{\mathbf{y}}\omega(\mathbf{y}) - d^*_{\mathbf{y}}\omega(\mathbf{y}) = 0.\end{aligned}$$

Since ω was an arbitrary test function, we must have that $E \equiv 0$ at $t = 0$. Thus by uniqueness, $E \equiv 0$ for all $t > 0$.

The proof of the second assertion follows in a similar manner. □

We will use this result to give an explicit formula for K_k in terms of $K_{k\pm 1}$.

Theorem 2.2. *Let M be an open, complete manifold with Ricci curvature bounded below. Then the differential forms heat kernel obey the following relation*:

$$K_k(\mathbf{x},\mathbf{y},t) = d_{\mathbf{x}}\, d_{\mathbf{y}} \int_t^\infty K_{k-1}(\mathbf{x},\mathbf{y},\tau)\, d\tau + d^*_{\mathbf{x}}\, d^*_{\mathbf{y}} \int_t^\infty K_{k+1}(\mathbf{x},\mathbf{y},\tau)\, d\tau.$$

Proof. Let K_k be the k-form heat kernel. Clearly,

$$K_k(\mathbf{x},\mathbf{y},t) = -\int_t^\infty \frac{\partial}{\partial\tau} K_k(\mathbf{x},\mathbf{y},\tau)\, d\tau,$$

since K_k tends to zero (pointwise) as t increases. Since K_k is a solution of the heat equation, we can replace the time derivative with $-\Delta_{\mathbf{x}} = -d_{\mathbf{x}} d^*_{\mathbf{x}} - d^*_{\mathbf{x}} d_{\mathbf{x}}$, so

$$K_k(\mathbf{x},\mathbf{y},t) = \int_t^\infty \big(d_{\mathbf{x}} d^*_{\mathbf{x}} + d^*_{\mathbf{x}} d_{\mathbf{x}}\big) K_k(\mathbf{x},\mathbf{y},\tau)\, d\tau.$$

Using Lemma 2.1, we can rewrite the above as

$$K_k(\mathbf{x},\mathbf{y},t) = \int_t^\infty d_{\mathbf{x}}\, d_{\mathbf{y}} K_{k-1}(\mathbf{x},\mathbf{y},\tau) + d^*_{\mathbf{x}}\, d^*_{\mathbf{y}} K_{k+1}(\mathbf{x},\mathbf{y},\tau)\, d\tau. \qquad \square$$