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The group of unimodular automorphisms of a principal bundle and the Euler–Yang–Mills equations

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ABSTRACT

Given a principal bundle $G \hookrightarrow P \to B$ (each being compact, connected and oriented) and a G-invariant metric h^P on P which induces a volume form μ^P, we consider the group of all unimodular automorphisms $\mathrm{SAut}(P,\mu^P) := \{\varphi \in \mathrm{Diff}(P) \mid \varphi^*\mu^P = \mu^P$ and φ is G-equivariant$\}$ of P, and determines its Euler equation à la Arnold. The resulting equations turn out to be (a particular case of) the Euler–Yang–Mills equations of an incompressible classical charged ideal fluid moving on B. It is also shown that the group $\mathrm{SAut}(P,\mu^P)$ is an extension of a certain volume preserving diffeomorphisms group of B by the gauge group $\mathrm{Gau}(P)$ of P.

1. Introduction

Since [3], it is well known that an appropriate configuration space for the study of equations of hydrodynamical type (more precisely, the incompressible Euler equations of an incompressible fluid) on a Riemannian manifold (M,g) endowed with a volume form μ (μ being not necessarily induced by the metric g), is given by the group of all unimodular diffeomorphisms $\mathrm{SDiff}(M,\mu) := \{\varphi \in \mathrm{Diff}(M) \mid \varphi^*\mu = \mu\}$ of M. This group is – in a suitably chosen sense – an infinite dimensional Lie group whose Lie algebra $\mathfrak{X}(M,\mu) := \{X \in \mathfrak{X}(M) \mid \mathrm{div}_\mu(X) = 0\}$ is the space of divergence free vector fields endowed with the opposite of the usual vector field bracket, and if $X \in \mathfrak{X}(M,\mu)$ is a time-dependant divergence free vector field describing the velocity field of an incompressible fluid, then its dynamics is governed by the incompressible Euler equation $\frac{d}{dt}X + \nabla_X X = \nabla p$, where p is the pressure of the fluid. It turns out that this equation characterizes geodesics on $\mathrm{SDiff}(M,\mu)$ with respect to the natural right-invariant L^2-metric on $\mathrm{SDiff}(M,\mu)$ (see [5]), and can be seen as an Euler equation (or Lie–Poisson equation) on the "regular dual" of $\mathfrak{X}(M,\mu)$ (see [2]).

In this paper, we propose another configuration space to study the Euler equation when some symmetries are involved. Our point of departure is to assume that the fluid evolves on the total space of a principal bundle $G \hookrightarrow P \to B$ (P being connected and oriented). We assume also that the metric h^P on P is G-invariant. In particular, the volume form μ^P on P

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induced by h^P is also G-invariant. This leads naturally to consider the group $\mathrm{SAut}(P, \mu^P)$ of automorphisms of P preserving the volume form μ^P instead of the group $\mathrm{SDiff}(P, \mu^P)$. In other words, we assume the vector field describing the velocity of the fluid to be initially G-invariant. This approach allows us to describe the Euler equation (in the presence of symmetries), as a system of two coupled equations, one living on the space of free divergence (for a certain volume form) vector fields on B, the other living on the Lie algebra of the gauge group $\mathrm{Gau}(P)$ of P. In some cases, these equations are a particular case of the Euler–Yang–Mills equation of an incompressible classical charged ideal fluid moving on B, and are physically relevant for the cases $G = S^1$ (super-conductivity equation, see [20]), $G = SU(2)$ and $G = SU(3)$ (chromohydrodynamics, see [7,6]). The terminology "Euler–Yang–Mills equation" comes from [6].

The second section of this paper describes the Lie group structure of the group $\mathrm{SDiff}(M, \mu)^G$ of all G-equivariant diffeomorphisms of a compact manifold M which preserve a volume form μ. The arguments are essentially those used by Hamilton in [8], Theorem 2.5.3, except that one has to check the constructions involving the Nash–Moser inverse function theorem to "respect symmetries". In Section 3, the careful study of the "structure" of a G-invariant volume form μ^P on the total space P of a principal bundle $G \hookrightarrow P \to B$, allows us to give an integration formula (Proposition 3.11) which is necessary to determine the Euler equation of the group $\mathrm{SAut}(P, \mu^P)$ (Theorem 4.19). Finally in Section 5, we show, in the same spirit of [1], that $\mathrm{SAut}(P, \mu^P)$ is a $\mathrm{Gau}(P)$-principal bundle whose base is a collection of connected components of $\mathrm{SDiff}(B, V\mu^B)$, where $V\mu^B$ is a volume form on B related to the volume of the orbits of P. In particular, $\mathrm{SAut}(P, \mu^P)$ is a non-abelian extension of this collection of connected components of $\mathrm{SDiff}(B, V\mu^B)$ by the gauge group $\mathrm{Gau}(P)$.

2. The group $\mathrm{SDiff}(M, \mu)^G$ as a tame Lie group

This section deals with the differentiable and Lie group structure of some subgroups of the group of smooth diffeomorphisms of a compact manifold, using the infinite dimensional geometry point of view. For that purpose, we will use the category of tame Fréchet manifolds developed by Hamilton in [8], and not simply the usual category of Fréchet manifolds.[2] This choice is motivated by the necessity to use an inverse function theorem, which is available in Hamitlon's category contrary to the general Fréchet setting.

For the convenience of the reader, we recall here the basic definitions relevant for Hamilton's category:

Definition 2.1.

(i) A graded Fréchet space $(F, \{\|.\|_n\}_{n\in\mathbb{N}})$, is a Fréchet space F whose topology is defined by a collection of seminorms $\{\|.\|_n\}_{n\in\mathbb{N}}$ which are increasing in strength:

$$\|x\|_0 \leqslant \|x\|_1 \leqslant \|x\|_2 \leqslant \cdots \tag{1}$$

for all $x \in F$.

(ii) A linear map $L : F \to G$ between two graded Fréchet spaces F and G is tame (of degree r and base b) if for all $n \geqslant b$, there exists a constant $C_n > 0$ such that for all $x \in F$,

$$\big\|L(x)\big\|_n \leqslant C_n \|x\|_{n+r}. \tag{2}$$

(iii) If $(B, \|.\|_B)$ is a Banach space, then $\Sigma(B)$ denotes the graded Fréchet space of all sequences $\{x_k\}_{k\in\mathbb{N}}$ of B such that for all $n \geqslant 0$,

$$\big\|\{x_k\}_{k\in\mathbb{N}}\big\|_n := \Sigma_{k=0}^{\infty} e^{nk} \|x_k\|_B < \infty. \tag{3}$$

(iv) A graded Fréchet space F is tame if there exist a Banach space B and two tame linear maps $i : F \to \Sigma(B)$ and $p : \Sigma(B) \to F$ such that $p \circ i$ is the identity on F.

(v) Let F, G be two tame Fréchet spaces, U an open subset of F and $f : U \to G$ a map. We say that f is a smooth tame map if f is smooth[3] and if for every $k \in \mathbb{N}$ and for every $(x, u_1, \ldots, u_k) \in U \times F \times \cdots \times F$, there exist a neighborhood V of $(x, u_1, \ldots, u_k)$ in $U \times F \times \cdots \times F$ and $b_k, r_0, \ldots, r_k \in \mathbb{N}$ such that for every $n \geqslant b_k$, there exists $C^V_{k,n} > 0$ such that

$$\big\|D^k f(y)\{v_1, \ldots, v_k\}\big\|_n \leqslant C^V_{k,n}\big(1 + \|y\|_{n+r_0} + \|v_1\|_{n+r_1} + \cdots + \|v_k\|_{n+r_k}\big), \tag{4}$$

for every $(y, v_1, \ldots, v_k) \in V$, where $D^k f : U \times F \times \cdots \times F \to G$ denotes the kth derivative of f.

Remark 2.2. In the sequel, we will use interchangeably the notation $(Df)(x)\{v\}$ or $f_{*_x} v$ for the first derivative of f at a point x in direction v.

[2] The reader should be aware that beyond the Banach case, several nonequivalent theories of infinite dimensional manifolds coexist (see [12]), but when the modelling spaces are Fréchet spaces, then most of these theories coincide, and it is thus natural to talk, without any further references, of a Fréchet manifold (as defined in [8] for example).

[3] By smooth we mean that $f : U \subseteq F \to G$ is continuous and that for all $k \in \mathbb{N}$, the kth derivative $D^k f : U \times F \times \cdots \times F \to G$ exists and is jointly continuous on the product space, such as described in [8].