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The area as a natural pseudo-Hermitian structure on the spaces of plane polygons and curves

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A B S T R A C T

This paper is mainly concerned with the space of Euclidean polygons in the plane and the space of curves in the plane. We show that the area function determines a pseudo-Hermitian structure on the space of orientation-preserving similarity classes of Euclidean plane polygons. The area as a pseudo-Hermitian structure on the space of orientation-preserving similarity classes of plane closed curves up to reparametrization is also explored. Explicit geodesics of such spaces are calculated.

1. Introduction

1.1. Area and shapes of polygons

There are remarkable examples of the use of Euclidean area of polygons to get geometric structures:

- Thurston constructed an $(n-3)$-dimensional complex hyperbolic structure on the space of orientation-preserving similarity classes of flat metrics on the sphere S^2 with n conical singularities of prescribed conical angles $\theta_1, \dots, \theta_n$, where $0 < \theta_j < 2\pi$ for all j. The condition $0 < \theta_j < 2\pi$ is important (see Example 12). The method of complex hyperbolization used by Thurston involves the area of polyhedra to obtain a pseudo-Hermitian form of signature $(1, n-3)$ (Proposition 3.3 in [1]).
- Bavard and Ghys considered the area function as a quadratic form on the space of similarity classes of plane polygons with prescribed interior angles $\theta_1, \dots, \theta_n$. They calculated the signature of this area form and showed that, in the case $0 < \theta_j < \pi$ for all j, the space of polygons has a real hyperbolic structure of dimension $n-3$ (Proposition 1 in [2], see also [3]).
- Veech studied the area pseudo-Hermitian form on the space of singular flat metrics on an arbitrary compact orientable surface, up to orientation-preserving similarity. In Section 14 of [4], Veech computed the signature of this area form as a function of the prescribed conical angles.
- Previously to all the above mentioned examples there is a famous article [5] by Deligne and Mostow. In this, Deligne and Mostow showed a very different aspect of the complex hyperbolization of the space of orientation-preserving similarity

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classes of singular flat metrics on the sphere, using techniques from algebraic geometry. In their paper, the area function appears as a pseudo-Hermitian form acting on some space of differential forms (Paragraphs 2.18–2.21 in [5]).

In this paper we use the area, in the same spirit as Thurston, to endow with a pseudo-Hermitian form the vector space of translation classes of plane Euclidean polygons with n vertices. As a consequence, the space of orientation-preserving similarity classes of Euclidean polygons with n vertices has a natural pseudo-Hermitian structure, and the real part of the pseudo-Hermitian structure gives a pseudo-Riemannian metric. This result generalizes the real hyperbolic structure for polygons of Bavard and Ghys, because we do not impose the interior angles restriction. Moreover, the Thurston's complex hyperbolic structure can be seen as the restriction of our pseudo-Hermitian form to a certain linear subspace (Remark 10), a fact that is not mentioned in [1]. We calculate the geodesics of the pseudo-Riemannian metric (Proposition 8). We also note, based on the work by Veech, that the space of hyperelliptic curves of given degree has a pseudo-Hermitian structure (Proposition 14), but such a geometric structure is not geodesically complete.

1.2. Area and shapes of curves

From the same perspective, we note that the area of the region bounded by a closed curve in the Euclidean plane determines a pseudo-Hermitian form on the vector space of translation classes of parametrized plane closed curves. We calculate the geodesic paths in the space of orientation-preserving similarity classes of parametrized plane closed curves (Proposition 18).

The space of shapes is, by definition, the space of orientation-preserving similarity classes of smooth plane closed curves up to orientation-preserving reparametrization. The space of shapes is an interesting object, and several Riemannian structures on it have been studied recently (see for example [6–8]). If one wishes to construct the "optimal" deformation between shapes, a natural formulation of "optimal" is simply to consider the geodesic path from one shape to another shape, with respect to some geometric structure on the space of shapes. That is why some papers make special emphasis on the description of geodesics (for example [9,10]), so that the concept of geodesic can be used to quantify the intuitive idea of whether two given shapes are similar, which is a main subject in computer vision and image recognition. We calculate the geodesics of the space of shapes endowed with the area pseudo-Hermitian form (Theorem 22).

2. The area pseudo-Hermitian form on the space of polygons

2.1. The area pseudo-Hermitian form

Let $n \geqslant 3$ be an integer. A polygon with n distinguished vertices in the Euclidean plane is determined by a point $(z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ where $z_1, z_2, \dots, z_n$ are its consecutive vertices. Any kind of degenerations and self-crossings are allowed. We say that a polygon is *singular* if it has self-crossings or repeated vertices, otherwise we call the polygon *simple*. For example, the polygons

$$\left(e^0, e^{4\pi i/5}, e^{8\pi i/5}, e^{2\pi i/5}, e^{6\pi i/5}\right), \qquad \left(e^0, e^{4\pi i/5}, e^{8\pi i/5}, e^{8\pi i/5}, e^{2\pi i/5}\right), \qquad \left(e^0, e^{4\pi i/5}, e^{6\pi i/5}, e^0, e^{6\pi i/5}\right)$$

are singular because the former has self-crossings and the other two have repeated vertices.

The *space of oriented polygonal shapes* $\mathcal{PS}(n)$ is the space of polygons with n distinguished vertices up to equivalence by orientation-preserving similarity; that is, two polygons $(z_1, \dots, z_n)$ and $(w_1, \dots, w_n)$ are identified if and only if there exists a complex affine transformation $f(z) = az + b$ of $\mathbb{C}$, $a \neq 0$, so that $f(z_k) = w_k$ for all $k = 1, \dots, n$. Up to a translation, each class of polygons has a representative $(z_1, \dots, z_n)$ such that $z_1 + \dots + z_n = 0$, simply translate the centroid of the vertices to the origin. If $V(n) \subset \mathbb{C}^n$ denotes the hyperplane $V(n) = \{(z_1, \dots, z_n)\colon\ z_1 + \dots + z_n = 0\}$, the space $\mathcal{PS}(n)$ can be identified with the complex projectivization $\mathbb{P}(V(n))$ of the vector space $V(n)$.

The signed area of a polygon $Z = (z_1, \dots, z_n)$ is

$$\mathcal{A}(Z) = \frac{i}{4} \sum_{j=1}^{n} (z_j \bar{z}_{j+1} - z_{j+1} \bar{z}_j), \tag{1}$$

where subscripts should be taken mod n. This is a natural quadratic form on the set of polygons $\mathbb{C}^n$. By polarization we get the *area pseudo-Hermitian form*

$$\big\langle (z_1, \dots, z_n), (w_1, \dots, w_n) \big\rangle_{\mathcal{A}} = \frac{i}{4} \sum_{j=1}^{n} (z_j \bar{w}_{j+1} - z_{j+1} \bar{w}_j). \tag{2}$$

Remark 1. It appears that the polygon $(z_1, \dots, z_n)$ has a special vertex z_1. However, note that the shift transformation

$$S(z_1, \dots, z_n) = (z_2, z_3, z_4, \dots, z_{n-1}, z_n, z_1)$$

is an isometry for the pseudo-Hermitian form $\langle \cdot, \cdot \rangle_{\mathcal{A}}$.