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A vanishing theorem on Kaehler Finsler manifolds ☆

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A R T I C L E I N F O

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A B S T R A C T

Let M be a connected compact complex manifold endowed with a strongly pseudoconvex complex Finsler metric F. In this paper, we first define the complex horizontal Laplacian $\Box_h$ and complex vertical Laplacian $\Box_v$ on the holomorphic tangent bundle $T^{1,0}M$ of M, and then we obtain a precise relationship among $\Box_h, \Box_v$ and the Hodge–Laplace operator $\triangle$ on $(T^{1,0}M, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle$ is the induced Hermitian metric on $T^{1,0}M$ by F. As an application, we prove a vanishing theorem of holomorphic p-forms on M under the condition that F is a Kaehler Finsler metric on M.

0. Introduction

The existence of holomorphic vector fields on complex manifolds reflects both the topological and geometrical information of the complex manifolds. In order to prove the non-existence of certain objects of geometric interest, such as holomorphic vector fields and harmonic forms on compact Kaehler manifolds, Bochner [10,11] initiated the now-called *Bochner technique* [25]. This technique involves an application of both the Laplacian and maximum principle on Riemannian manifolds. Along this line, we refer to [16,23,25] and [26].

However, the most often used intrinsic metrics on complex manifolds are usually complex Finsler metrics, such as the Kobayashi and Carathéodory metrics. They arise naturally in the geometric theory of several complex variables, see [1,18] and the recent work of [24].

Let M be a connected complex manifold endowed with a strongly pseudoconvex complex Finsler metric F in the sense of [1]. In this paper, we introduce the complex horizontal Laplacian $\Box_h$ and complex vertical Laplacian $\Box_v$ on the holomorphic tangent bundle $T^{1,0}M$ of M, establish a precise relationship among $\Box_h, \Box_v$ and the Hodge–Laplace operator $\triangle$ on $T^{1,0}M$. As an application, we prove a vanishing theorem of holomorphic p-forms on M under the condition that F is a Kaehler Finsler metric on M.

However, there is no canonical way to define Laplacian on Finsler manifolds, we refer to [2,8] and the recently work of [21]. Our motivation for introducing $\Box_h$ and $\Box_v$ comes from the recent study of vanishing theorem of cohomology on the strongly pseudoconvex complex Finsler manifold (M, F), see [3–7,13–15,19–21,24]. The method developed in this paper seems flexible and also more in spirit of Bochner [26].

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It is known to us that if F comes from a Kaehler metric on M, then the complex Laplacian $\Box$ on M and the Hodge–Laplace operator $\triangle$ on M satisfy $\triangle = 2\Box$ and $\Box = \overline{\Box}$, see [17]. One may seek the same relationship among $\Box_h, \Box_v$ and the Hodge–Laplace operator $\triangle$ on $T^{1,0}M$. But in general there is no such relationship among $\Box_h, \Box_v$ and $\triangle$ even F is a Kaehler Finsler metric on M. We also note that the Cartan connection associated to a strongly pseudoconvex complex Finsler manifold (M, F) is different from the Chern Finsler connection associated to (M, F), and their difference is not a tensor [1]. These differences motivate us to investigate the vanishing theorem on a strongly pseudoconvex complex Finsler manifold (M, F) via the method of complex differential geometry, which also proves to be effective in the treatment of complex Finsler submanifolds [29].

The key point is that the complex horizontal Laplacian $\Box_h$ also lives in the projectivized tangent bundle $\mathbb{P}\tilde{M}$, which is a compact complex manifold whenever M is compact. This makes it possible for us to establish a divergence type lemma and furthermore use it obtain a vanishing theorem of holomorphic p-forms on M when F is a Kaehler Finsler metric on M.

The main results in this paper are (cf. Theorem 5.5, Theorem 5.7 and Theorem 6.3).

Theorem A. *Let (M, F) be a Kaehler Finsler manifold with the associated complex Rund connection D, and $\Omega(\cdot,\cdot)$ be the curvature tensor of D. Then*

$$\Box_h = -G^{\bar{\beta}\alpha} D_{\delta_{\bar{\beta}}} D_{\delta_\alpha} - G^{\bar{\beta}\alpha} e\big(dz^\gamma\big) i(\delta_\alpha)\big\{[\delta_\gamma, \delta_{\bar{\beta}}] + \Omega(\delta_\gamma, \delta_{\bar{\beta}})\big\}. \tag{0.1}$$

Theorem B. *Let (M, F) be a strongly pseudoconvex complex Finsler manifold with the associated complex Rund connection D, and $\langle\cdot,\cdot\rangle$ be the induced Hermitian metric on $T^{1,0}M$ by F. Then the Laplace–Beltrami operator $\triangle$ associated to $(T^{1,0}M, \langle\cdot,\cdot\rangle)$, the complex horizontal Laplacian $\Box_h$ and the complex vertical Laplacian $\Box_v$ associated to (M, F) are related by*

$$\triangle f = \Box_h f + \overline{\Box}_h f + \Box_v f + \overline{\Box}_v f, \quad \forall f \in C^\infty(\tilde{M}). \tag{0.2}$$

Theorem C. *Let (M, F) be a compact Kaehler Finsler manifold. If φ is a holomorphic p-form $(p \geqslant 1)$ on M such that*

$$\mathrm{Re}\big\{G^{\bar{\beta}\alpha}\big\langle\Omega(\delta_\alpha, \delta_{\bar{\beta}})\varphi, \varphi\big\rangle\big\} \leqslant 0, \tag{0.3}$$

then it must be $|\mathcal{D}_h\varphi|^2 = 0$, i.e., $\mathcal{D}_h\varphi = 0$. If furthermore

$$\mathrm{Re}\big\{G^{\bar{\beta}\alpha}\big\langle\Omega(\delta_\alpha, \delta_{\bar{\beta}})\varphi, \varphi\big\rangle\big\} < 0, \tag{0.4}$$

then no such holomorphic p-form φ exists on M.

This paper is arranged as follows. In Section 1 we recall some basic fact about the complex Rund connection D associated to (M, F). In Section 2, the exterior derivative operator d on $T^{1,0}M$ is investigated, and a decomposition of d into four endomorphisms $\mathcal{D}_h, \mathcal{D}_v, \mathcal{D}_3, \mathcal{D}_4$ and their conjugations are obtained. In Section 3, as in Hermitian geometry [17], we introduce a global Hermitian inner product $(\cdot,\cdot)$ in $\mathscr{A}_c^{p,q;r,s}(\tilde{M})$, where $\mathscr{A}_c^{p,q;r,s}(\tilde{M})$ is the space of $(p,q;r,s)$-forms with compact support in $\tilde{M}$. Then we define a Hodge star operator $*: \mathscr{A}_c^{p,q;r,s}(\tilde{M}) \to \mathscr{A}_c^{p,q;r,s}(\tilde{M})$ and the metric adjoints $\mathcal{D}_h^*, \mathcal{D}_v^*, \mathcal{D}_3^*, \mathcal{D}_4^*$ and their conjugations with respect to the global inner product $(\cdot,\cdot)$. In Section 4, the Levi-Civita connection $\widetilde{\nabla}$ associated to $(T^{1,0}M, \langle,\rangle)$ is expressed in terms of the connection coefficients of the complex Rund connection. In Section 5, we obtain the global form of $\Box_h$ and $\Box_v$. In Section 6 we establish a divergence type lemma and prove a vanishing theorem of holomorphic p-forms on Kaehler Finsler manifolds.

1. Complex Rund connection

In this section we recall the complex Rund connection associated to (M, F). The basic reference to this section is [1,22].

Let M be a connected complex manifold of $\dim_{\mathbb{C}} M = n$. Denote by $\pi: T^{1,0}M \to M$ the holomorphic tangent bundle of M, and set $\tilde{M} = T^{1,0}M \backslash \{o\}$, where o is the zero section of $T^{1,0}M$. Obviously, both $T^{1,0}M$ and $\tilde{M}$ are non-compact complex manifolds.

Let $\{z^\alpha\}$ be the local holomorphic coordinates on M, and $\{z^\alpha, v^\alpha\}$ be the induced local holomorphic coordinates on $T^{1,0}M$. Set

$$\partial_\alpha := \frac{\partial}{\partial z^\alpha}, \qquad \dot{\partial}_\alpha := \frac{\partial}{\partial v^\alpha}.$$

In this paper, M is assumed to be endowed with a strongly pseudoconvex complex Finsler metric F in the sense that $F: T^{1,0}M \to [0, +\infty)$ satisfying

(1) $G = F^2(z, v)$ is smooth on $\tilde{M}$;
(2) $F(z, v) > 0$ for all $(z, v) \in \tilde{M}$;
(3) $F(z, \lambda v) = |\lambda| F(z, v)$ for all $(z, v) \in T^{1,0}M$ and $\lambda \in \mathbb{C}$;
(4) the Levi matrix

$$(G_{\alpha\bar{\beta}}) := (\dot{\partial}_\alpha \dot{\partial}_{\bar{\beta}} G) \tag{1.1}$$

is positive definite on $\tilde{M}$.