Available online at www.sciencedirect.com
ScienceDirect
Differential Geometry and its Applications 26 (2008) 553–565
DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
www.elsevier.com/locate/difgeo

Classification of 1st order symplectic spinor operators over contact projective geometries

Svatopluk Krýsl [a,b,*]

[a] *Charles University, Sokolovská 83, Praha, Czech Republic*
[b] *Humboldt-Universität zu Berlin, Unter den Linden 6, Berlin, Germany*

Received 16 October 2006; received in revised form 20 May 2007

Available online 19 February 2008

Communicated by J. Slovak

Abstract

We give a classification of 1st order invariant differential operators acting between sections of certain bundles associated to Cartan geometries of the so-called metaplectic contact projective type. These bundles are associated via representations, which are derived from the so-called higher symplectic (sometimes also called harmonic or generalized Kostant) spinor modules. Higher symplectic spinor modules are arising from the Segal–Shale–Weil representation of the metaplectic group by tensoring it by finite dimensional modules. We show that for all pairs of the considered bundles, there is at most one 1st order invariant differential operator up to a complex multiple and give an equivalence condition for the existence of such an operator. Contact projective analogues of the well known Dirac, twistor and Rarita–Schwinger operators appearing in Riemannian geometry are special examples of these operators.

MSC: 22E46; 58J60; 58J70

Keywords: Metaplectic contact projective geometry; Symplectic spinors; Segal–Shale–Weil representation; Kostant spinors; First order invariant differential operators

1. Introduction

The operators we would like to classify are 1st order invariant differential operators acting between sections of vector bundles associated to metaplectic contact projective geometries via certain minimal globalizations.

Metaplectic contact projective geometry on an odd dimensional manifold is first a contact geometry, i.e., it is given by a corank one subbundle of the tangent bundle of the manifold which is nonintegrable in the Frobenius sense in each point of the manifold. Second part of the metaplectic contact projective structure on a manifold is given by a class of projectively equivalent contact partial affine connections. Here, partial contact means that the connections are compatible with the contact structure and that they are acting only on the sections of the contact subbundle. These

* Correspondence at: Charles University, Sokolovská 83, Praha, Czech Republic.
E-mail address: krysl@karlin.mff.cuni.cz.

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2007.11.037

connections are called projectively equivalent because they have the same class of unparameterized geodesics going in the contact subbundle direction, see, e.g., D. Fox [9], where you can find a relationship between the contact projective geometries and classical path geometries. The adjective "metaplectic" suggests that in addition to contact projective geometries, the metaplectic contact projective structures include some spin phenomena like the spin structures over Riemannian manifolds. Metaplectic contact projective and contact projective geometries have their description also via Cartan geometries. Contact projective geometries could be modeled on a $(2l+1)$-dimensional projective space $\mathbb{PV}$ of a $(2l+2)$-dimensional real symplectic vector space $\mathbb{V}$, which we suppose to be equipped with a symplectic form ω. Here, the projective space is considered as a homogeneous space G/P, where G is the symplectic Lie group $Sp(\mathbb{V},\omega)$ acting transitively on $\mathbb{PV}$ by the factorization of its defining representation (on $\mathbb{V}$), and P is an isotropy subgroup of this action. In this case, it is easy to see that P is a parabolic subgroup, which turns out to be crucial for our classification. Contact projective geometry, in the sense of É. Cartan, are curved versions $(p:\mathcal{G}\to M,\omega)$ of this homogeneous (also called Klein) model G/P. There exist certain conditions (known as normalization conditions) under which the Cartan's principal bundle approach and the classical one (via the class of connections and the contact subbundle) are equivalent, see, e.g., Čap, Schichl [4] for details. We also remind that contact geometries are an arena for time-dependent Hamiltonian mechanics. Klein model of the metaplectic contact projective geometry consists of two groups $\tilde{G}$ and $\tilde{P}$, where $\tilde{G}$ is the metaplectic group $Mp(\mathbb{V},\omega)$, i.e., a nontrivial double covering of the symplectic group G, and $\tilde{P}$ is the preimage of P by this covering.

Symplectic spinor operators over projective contact geometries are acting between sections of the so-called higher symplectic spinor bundles. These bundles are associated via certain infinite dimensional irreducible admissible representations of the parabolic principal group P. The parabolic group P acts then nontrivially only by its Levi factor G_0, while the action of the unipotent part is trivial. The semisimple part $\mathfrak{g}_0^{ss}$ of the Lie algebra of the Levi part of the parabolic group P is isomorphic to the symplectic Lie algebra $\mathfrak{sp}(2l,\mathbb{R})$. Thus to give an admissible representation of P, we have to specify a representation of $\mathfrak{g}_0^{ss}$. Let us recall that the classification of first order invariant operators was done by Slovák, Souček in [24] (generalizing an approach of Fegan in [8]) for all finite dimensional irreducible representations and general parabolic subgroup P of a semisimple G (almost Hermitian structures are studied in detail). Nevertheless, there are some interesting infinite dimensional representations of the complex symplectic Lie algebra, to which we shall focus our attention. These representations form a class consisting of infinite dimensional modules with bounded multiplicities. Modules with bounded multiplicities are representations, for which there is a nonnegative integer, such that the dimension of each weight space of this module is bounded by it from above. Britten, Hooper and Lemire in [2] and Britten, Hooper in [3] showed that each of these modules appear as direct summands in a tensor product of a finite dimensional $\mathfrak{sp}(2l,\mathbb{C})$-module and the so-called Kostant (or basic) symplectic spinor module $\mathbb{S}_+$ and vice versa. Irreducible representations in this completely reducible tensor product are called higher symplectic, harmonic or generalized Kostant spinors. It is well known, that all finite dimensional modules over complex symplectic Lie algebra appear as irreducible submodules of a tensor power of the defining representation. Thus the infinite dimensional modules with bounded multiplicities are analogous to the spinor–vector representations of complex orthogonal Lie algebras. Namely, each finite dimensional module over orthogonal Lie algebra is an irreducible summand in the tensor product of a basic spinor representation and some power of the defining module (spinor–vector representations), or in the power of the defining representation itself (vector representations). In order to have a complete picture, it remains to show that the basic (or Kostant) spinors are analogous to the orthogonal ones, even though infinite dimensional. The basic symplectic spinor module $\mathbb{S}_+$ was discovered by Bertram Kostant (see [20]), when he was introducing half-forms for metaplectic structures over symplectic manifolds in the context of geometric quantization. While in the orthogonal case spinor representations can be realized using the exterior algebra of a maximal isotropic vector space, the symplectic spinor representations are realized using the symmetric algebra of certain maximal isotropic vector space (called Lagrangian in the symplectic setting). This procedure goes roughly as follows: one takes the Chevalley realization of the symplectic Lie algebra C_l by polynomial coefficients linear differential operators acting on polynomials $\mathbb{C}[z^1,\dots,z^l]$ in l complex variables. The space of polynomials splits into two irreducible summands over the symplectic Lie algebra, namely into the two basic symplectic spinor modules $\mathbb{S}_+$ and $\mathbb{S}_-$. There is a relationship between the modules $\mathbb{S}_+$ and $\mathbb{S}_-$ and the Segal–Shale–Weil or oscillator representation. Namely, the underlying C_l-structure of the Segal–Shale–Weil representation is isomorphic to $\mathbb{S}_+\oplus\mathbb{S}_-$.

In order to classify 1st order invariant differential operators, one needs to understand the structure of the space of P-homomorphisms between the so called 1st jets prolongation P-module of the domain module and the target representation of P, see Section 4. Thus the classification problem translates into an algebraic one. In our case, rep-