Differential Geometry and its Applications 27 (2009) 793–808

Contents lists available at ScienceDirect

Differential Geometry and its Applications

www.elsevier.com/locate/difgeo

Drapeau theorem for differential systems

Kazuhiro Shibuya, Keizo Yamaguchi *

Department of Mathematics, Graduate School of Science, Hokkaido University, Sapporo 060-0810, Japan

A R T I C L E I N F O

Article history:
Received 18 July 2008
Received in revised form 29 December 2008
Available online 7 April 2009
Communicated by J. Slovák

MSC:
53C15
58A15
58A20
58A30

Keywords:
Goursat flag
m-flag of length k
Rank 1 prolongation

A B S T R A C T

Generalizing the theorem for Goursat flags, we will characterize those flags which are obtained by "rank 1 prolongation" from the space of 1 jets for 1 independent and m dependent variables.

1. Introduction

This paper is concerned with the Drapeau theorem for differential systems. By a differential system (R, D), we mean a distribution D on a manifold R, i.e., D is a subbundle of the tangent bundle $T(R)$. The *derived system* ∂D of D is defined, in terms of sections, by

$$\partial\mathcal{D} = \mathcal{D} + [\mathcal{D}, \mathcal{D}],$$

where $\mathcal{D} = \Gamma(D)$ denotes the space of sections of D. In general ∂D is obtained as a subsheaf of the tangent sheaf of R (for the precise argument, see e.g. [15,3]). Moreover higher derived systems $\partial^i D$ are defined successively by

$$\partial^i D = \partial(\partial^{i-1} D),$$

where we put $\partial^0 D = D$ by convention. We also define the i-th weak derived system $\partial^{(i)} D$ of D inductively by

$$\partial^{(i)}\mathcal{D} = \partial^{(i-1)}\mathcal{D} + [\mathcal{D}, \partial^{(i-1)}\mathcal{D}],$$

where $\partial^{(0)} D = D$ and $\partial^{(i)}\mathcal{D}$ denotes the space of sections of $\partial^{(i)} D$. In this paper, a differential system (R, D) is called regular if $\partial^i D$ are subbundles of $T(M)$ for every $i \geqslant 1$.

We say that (R, D) is an m-flag of length k, if (R, D) is regular and has a derived length k, i.e., $\partial^k D = T(R)$;

$$D \subset \partial D \subset \cdots \subset \partial^{k-2} D \subset \partial^{k-1} D \subset \partial^k D = T(R),$$

* Corresponding author.
E-mail address: yamaguch@math.sci.hokudai.ac.jp (K. Yamaguchi).

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2009.03.017

such that $\operatorname{rank} D = m+1$ and $\operatorname{rank} \partial^i D = \operatorname{rank} \partial^{i-1} D + m$ for $i = 1, \dots, k$. In particular $\dim R = (k+1)m + 1$.

Especially (R, D) is called a *Goursat flag* (un drapeau de Goursat) of length k when $m = 1$. Historically, by Engel, Goursat and Cartan, it is known that a Goursat flag (R, D) of length k is locally isomorphic, at a generic point, to the canonical system $(J^k(M,1), C^k)$ on the k-jet spaces of 1 independent and 1 dependent variable (for the definition of the canonical system $(J^k(M,1), C^k)$, see Section 2). The characterization of the canonical (contact) systems on jet spaces was given by R. Bryant in [2] for the first order systems and in [15] and [16] for higher order systems for n independent and m dependent variables. However, it was first explicitly exhibited by A. Giaro, A. Kumpera and C. Ruiz in [6] that a Goursat flag of length 3 has singularities and the research of singularities of Goursat flags of length k $(k \geqslant 3)$ began as in [9]. To this situation, R. Montgomery and M. Zhitomirskii constructed the "Monster Goursat manifold" by successive applications of the "Cartan prolongation of rank 2 distributions [4]" to a surface and showed that every germ of a Goursat flag (R, D) of length k appears in this "Monster Goursat manifold" in [8], by first exhibiting the following Sandwich Lemma for (R, D);

$$\begin{array}{ccccccccc} D & \subset & \partial D & \subset \cdots \subset & \partial^{k-2} D & \subset & \partial^{k-1} D & \subset & \partial^k D = T(R) \\ \cup & & \cup & & \cup & & & & \\ \mathrm{Ch}(D) & \subset & \mathrm{Ch}(\partial D) & \subset \mathrm{Ch}(\partial^2 D) \subset \cdots \subset & \mathrm{Ch}(\partial^{k-1} D) & & & & \end{array}$$

where $\mathrm{Ch}(\partial^i D)$ is the Cauchy characteristic system of $\partial^i D$ and $\mathrm{Ch}(\partial^i D)$ is a subbundle of $\partial^{i-1} D$ of corank 1 for $i = 1, \dots, k-1$. Here the *Cauchy Characteristic System* $\mathrm{Ch}(C)$ of a differential system (R, C) is defined by

$$\mathrm{Ch}(C)(x) = \bigl\{ X \in C(x) \mid X \rfloor d\omega_i \equiv 0 \pmod{\omega_1, \dots, \omega_s} \text{ for } i = 1, \dots, s \bigr\},$$

where $C = \{\omega_1 = \cdots = \omega_s = 0\}$ is defined locally by defining 1-forms $\{\omega_1, \dots, \omega_s\}$. Moreover, after [8], P. Mormul defined the notion of a *special m-flag* of length k for $m \geqslant 2$ to characterize those m-flags which are obtained by successive applications of the "generalized Cartan prolongation" to the space of 1-jets of 1 independent and m dependent variables.

The main purpose of this paper is first to clarify the procedure of "rank 1 prolongation" of an arbitrary differential system (R, D) of rank $m+1$, and to give good criteria for an m-flag of length k to be special, i.e., to be locally isomorphic to the k-th rank 1 prolongation $(P^k(M), C^k)$ of a manifold M of dimension $m+1$ (by construction, $P^k(M)$ contains $J^k(M,1)$ as an open dense subset. See Section 3). More precisely we will show for an m-flag of length k for $m \geqslant 2$;

Drapeau Theorem. *Let (R, D) be an m-flag of length k. If $m \geqslant 2$, then the following statements are equivalent*:

(i) *(R, D) is locally isomorphic to $(P^k(M), C^k)$.*
(ii) *There exists a completely integrable subbundle F of $\partial^{k-1} D$ of corank* 1 (*see the sentence following Proposition* 4.1 *for what F is in the case of $P^k(M)$*).
(iii) *(R, D) is a special m-flag* (*see Definition in Section* 4). *If $m \geqslant 3$, then* (i), (ii) *and* (iii) *are also equivalent to the following statement*
(iv) *$\partial^{k-1} D$ has Cartan rank* 1. *Finally, if $m \geqslant 4$, then* (i), (ii), (iii) *and* (iv) *are equivalent to the following statement*
(v) *$\partial^{k-1} D$ has Engel rank* 1.

Here, the *Cartan rank* of (R, C) is the smallest integer ρ such that there exist 1-forms $\{\pi^1, \dots, \pi^\rho\}$, which are independent modulo $\{\omega_1, \dots, \omega_s\}$ and satisfy

$$d\alpha \wedge \pi^1 \wedge \cdots \wedge \pi^\rho \equiv 0 \pmod{\omega_1, \dots, \omega_s} \quad \text{for } \forall \alpha \in \mathcal{C}^\perp = \Gamma(C^\perp),$$

where $C = \{\omega_1 = \cdots = \omega_s = 0\}$ and $C^\perp$ is the annihilator subbundle in $T^*(R)$ of C defined by $C^\perp(x) = \langle \{(\omega_1)_x, \dots, (\omega_s)_x\} \rangle \subset T^*_x(R)$ at each $x \in R$. Furthermore the *Engel* (*half*) *rank* of (R, C) is the smallest integer ρ such that

$$(d\alpha)^{\rho+1} \equiv 0 \pmod{\omega_1, \dots, \omega_s} \quad \text{for } \forall \alpha \in \mathcal{C}^\perp.$$

Obviously, if (R, C) has Cartan rank ρ, then (R, C) has Engel rank less than ρ (cf. II Section 4 in [3]).

For this purpose, we will first review the geometric construction of jet spaces in Section 2 and clarify the procedure of rank 1 prolongation in Section 3. In Section 4, we will analyze the notion of a special m-flag of length k and reestablish the local characterization of $(P^k(M), C^k)$ by utilizing the Realization Lemma [15], which proves the equivalence of (i) and (iii) in the above *theorem*. In Section 5, we will show the equivalence of (iii) and (iv) or (v). In Section 6, we will show the equivalence of (iii) and (ii) and establish the above criteria (the *Drapeau Theorem*) for an m-flag of length k. Finally in Section 7, we will characterize the regular part $J^k(M,1)$ of $P^k(M)$ by the generating condition for the weak derived systems.

2. Geometric construction of jet spaces

In this section, we will briefly recall the geometric construction of jet bundles in general, following [15] and [16], which is our basis for the later considerations.

Let M be a manifold of dimension $m+n$. Fixing the number n, we form the space of n-dimensional *contact elements* to M, i.e., the Grassmann bundle $J(M, n)$ over M consisting of n-dimensional subspaces of tangent spaces to M. Namely, $J(M, n)$ is defined by

$$J(M, n) = \bigcup_{x \in M} J_x, \quad J_x = \mathrm{Gr}\bigl(T_x(M), n\bigr),$$