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Harmonic morphisms from the compact semisimple Lie groups and their non-compact duals

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Abstract

In this paper we prove the local existence of complex-valued harmonic morphisms from any compact semisimple Lie group and their non-compact duals. These include all Riemannian symmetric spaces of types II and IV. We produce a variety of concrete harmonic morphisms from the classical compact simple Lie groups $\mathbf{SO}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$ and globally defined solutions on their non-compact duals $\mathbf{SO}(n,\mathbb{C})/\mathbf{SO}(n)$, $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$ and $\mathbf{Sp}(n,\mathbb{C})/\mathbf{Sp}(n)$.

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1. Introduction

Harmonic morphisms between Riemannian manifolds are harmonic maps which satisfy an additional conformality condition, called horizontal (weak) conformality. As their fibres are often minimal, they are useful tools for the construction of minimal submanifolds. However, the two conditions imposed on them constitute an over-determined non-linear system of partial differential equations. For this reason they can be difficult to find and have no general existence theory, not even locally. On the contrary, most metrics on a 3-dimensional domain M^3 do not allow any local solutions with values in a surface N^2, see [5]. This makes it interesting to find geometric and topological conditions on the manifolds (M,g) and (N,h) which ensure the existence of solutions to the problem. For the general theory of harmonic morphisms between Riemannian manifolds we refer to the excellent book [4] and the regularly updated on-line bibliography [7].

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For harmonic morphisms $\phi:(M,g)\to(N,h)$ to exist, it is an advantage if the target manifold N is a surface, i.e. of dimension 2. In this case the problem is invariant under conformal changes of the metric on N^2. Hence, at least for local studies, the codomain can be assumed to be the standard complex plane.

It is known that in several cases, when the domain (M,g) is an irreducible Riemannian symmetric space, there exist complex-valued solutions to the problem, see for example [8,9,13], where the authors present the following conjecture.

Conjecture 1.1. *Let (M^m,g) be an irreducible Riemannian symmetric space of dimension $m\geqslant 2$. For each point $p\in M$ there exists a complex-valued harmonic morphism $\phi:U\to\mathbb{C}$ defined on an open neighbourhood U of p. If the space (M,g) is of non-compact type then the domain U can be chosen to be the whole of M.*

The conjecture is known to be true for the irreducible Riemannian symmetric spaces

$$\mathbf{SO}(p+q)/\mathbf{SO}(p)\times\mathbf{SO}(q),\qquad \mathbf{SO}_0(p,q)/\mathbf{SO}(p)\times\mathbf{SO}(q)$$

when $p\notin\{q,q\pm 1\}$,

$$\mathbf{SU}(p+q)/\mathbf{S}\big(\mathbf{U}(p)\times\mathbf{U}(q)\big),\qquad \mathbf{SU}(p,q)/\mathbf{S}\big(\mathbf{U}(p)\times\mathbf{U}(q)\big)$$

for any positive integers p,q, and for

$$\mathbf{Sp}(p+q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q),\qquad \mathbf{Sp}(p,q)/\mathbf{Sp}(p)\times\mathbf{Sp}(q)$$

when $p\neq q$, see [9].

In this paper we continue our study of complex-valued harmonic morphisms from Riemannian symmetric spaces. We prove the following existence theorem, see Section 9. For the definition of an orthogonal harmonic family, see the next section.

Theorem 1.2. *Let G be a compact semisimple Lie group equipped with a bi-invariant metric g and let $G^{\mathbb{C}}/G$ be its non-compact dual space.*

(i) *There exists an open and dense subset W^* of G and an orthogonal harmonic family $\mathcal{F}^*$ on W^*.*
(ii) *There exists an open subset W of $G^{\mathbb{C}}/G$ and an orthogonal harmonic family $\mathcal{F}$ on W.*
(iii) *If there exists a parabolic subgroup P of $G^{\mathbb{C}}$ such that the quotient $G^{\mathbb{C}}/P$ is a Hermitian symmetric space, then there is a globally defined orthogonal harmonic family $\mathcal{F}$ on $G^{\mathbb{C}}/G$.*

The collections of compact semisimple Lie groups and their non-compact duals include the irreducible Riemannian symmetric spaces of type II and IV, respectively. This means that by Theorem 1.2 we prove Conjecture 1.1 for the compact irreducible Riemannian symmetric spaces

$$\mathbf{SO}(n),\qquad \mathbf{SU}(n),\qquad \mathbf{Sp}(n),\qquad E_6,\qquad E_7,\qquad E_8,\qquad F_4,\qquad G_2$$

of type II, and the non-compact

$$\mathbf{SO}(n,\mathbb{C})/\mathbf{SO}(n),\qquad \mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n),\qquad \mathbf{Sp}(n,\mathbb{C})/\mathbf{Sp}(n),\qquad E_6^{\mathbb{C}}/E_6,\qquad E_7^{\mathbb{C}}/E_7$$

of type IV.

Leading up to the general existence theory we produce a variety of *concrete* complex-valued harmonic morphisms on the irreducible Riemannian symmetric spaces $\mathbf{SO}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$ of type II and $\mathbf{SO}(n,\mathbb{C})/\mathbf{SO}(n)$, $\mathbf{SL}_n(\mathbb{C})/\mathbf{SU}(n)$, $\mathbf{Sp}(n,\mathbb{C})/\mathbf{Sp}(n)$ of type IV.

Throughout this article we assume that all our manifolds are connected. Further we suppose that all our objects such as manifolds, maps etc. are smooth, i.e. in the C^∞-category. For our notation concerning Lie groups we refer to the comprehensive book [12].