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Natural differential operators and graph complexes ☆

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ABSTRACT

We show how the machine invented by S. Merkulov [S.A. Merkulov, Operads, deformation theory and F-manifolds, in: Frobenius Manifolds, Aspects Math., vol. E36, Vieweg, Wiesbaden, 2004, pp. 213–251; S.A. Merkulov, PROP profile of deformation quantization, Preprint, math.QA/0412257, December 2004; S.A. Merkulov, PROP profile of Poisson geometry, Comm. Math. Phys. 262 (1) (February 2006) 117–135] can be used to study and classify natural operators in differential geometry. We also give an interpretation of graph complexes arising in this context in terms of representation theory. As application, we prove several results on classification of natural operators acting on vector fields and connections.

Introduction

This work started in an attempt to understand S. Merkulov's idea of "PROP profiles" [18,21] and see if and how it may be used to investigate natural structures in geometry. It turned out that classifications of these geometric structures in many interesting cases boiled down to calculations of the cohomology of certain graph complexes. More precisely, for a wide class of natural operators, the following principle holds.

Principle. For a given type of natural differential operators, there exists a graph cochain complex $(\mathcal{G}r^*, \delta) = (\mathcal{G}r^0 \stackrel{\delta}{\to} \mathcal{G}r^1 \stackrel{\delta}{\to} \mathcal{G}r^2 \stackrel{\delta}{\to} \cdots)$ such that, in stable ranges,

$$\{\text{natural operators of a given type}\} \cong H^0(\mathcal{G}r_*, \delta).$$

Stability means that the dimension of the underlying manifold is bigger than some constant explicitly determined by the type of operators. For example, for multilinear natural operators $TM^{\times d} \to TM$ from the d-fold product of the tangent bundle into itself the stability means that $\dim(M) \geqslant d$. In smaller dimensions, "exotic" operations described in [5] occur.

In all cases we studied, the corresponding graph complex appeared to be acyclic in positive dimensions, so the cohomology describing natural operators was the only nontrivial piece of the cohomology of $(\mathcal{G}r_*, \delta)$. Standard philosophy of strongly homotopy structures [12] suggests that the graph complex $(\mathcal{G}r^*, \delta)$ describes *stable strongly homotopy operators* of a given type.

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Graph complexes arising in the Principle are in fact isomorphic to subspaces of fixed elements in suitable Chevalley–Eilenberg complexes, so, formally speaking, we claim that a certain Chevalley–Eilenberg cohomology is the cohomology of some graph complex. Instances of this phenomenon were systematically used by M. Kontsevich in his seminal paper [9]. The details of operadic graph complexes were then written down by J. Conant [2], J. Conant and K. Vogtmann [3,4], M. Mulase and M. Penkava [22], M. Penkava [24], and M. Penkava and A. Schwarz [25]. What makes the Principle exciting is the miraculous fact that the corresponding graph complexes are of the type studied during the "renaissance of operads" and powerful methods developed in this period culminating in [15,17,20] apply.

Another way to view the proposed method is as a formalization of the "abstract tensor calculus" attributed to R. Penrose. When we studied differential geometry in kindergarten, many of us, trying to avoid dozens of indices, drew simple pictures consisting of nodes representing tensors (which resembled little insects) and lines joining legs of these insects symbolizing contraction of indices. We attempt to put this kindergarten approach on a solid footing.

Thus the purpose of this paper is two-fold. The first one is to set up principles of abstract tensor calculus as a useful language for 'stable' geometric objects. This will be done in Sections 1–4. The logical continuation should be translating textbooks on differential geometry into this language, because all basic properties of fundamental objects (vector fields, forms, currents, connections and their torsions and curvatures) are of stable nature.

We then show, in Sections 5–7, how results on graph complexes may give explicit classifications of natural operators in stable ranges. As an example we derive from a rather deep result of [14] a characterization of operators on vector fields (Theorem 5.1 and its Corollary 5.3). As another application we prove that all natural operators on linear connections and vector fields, with values in vector fields, are freely generated by compositions of covariant derivatives and Lie brackets, and by traces of these compositions—see Theorems 7.2 and 7.6, and their Corollaries 7.3 and 7.7, in conjunction with Theorems 6.2 and 6.3.

This article is supplemented by [11] in which we explain the relation between invariant tensors and graphs. We believe that [11], which can be read independently, will help to understand the constructions of Sections 3 and 4.

The theory of invariant operators sketched out in this paper leads to directed, not necessarily connected or simply-connected, graphs. A similar theory can be formulated also for symplectic manifolds, where the corresponding graph complexes would be those appearing in the context of anti-modular operads (modular versions of anticyclic operads, see [16, Definition 5.20]). Something close to a symplectic version of our theory has in fact already been worked out in [27].

1. Natural operators

Informally, a natural differential operator is a recipe that constructs from a geometric object another one, in a natural fashion, and which is locally a function of coordinates and their derivatives.

1.1. Example. Let M be a n-dimensional smooth manifold. The classical *Lie bracket* $X, Y \mapsto [X, Y]$ is a natural operation that constructs from two vector fields on M a third one. Given a local coordinate system $(x^1, \dots, x^n)$ on M, the vector fields X and Y are locally expressions $X = \sum_{1\leqslant i\leqslant n} X^i \partial/\partial x^i$, $Y = \sum_{1\leqslant i\leqslant n} Y^i \partial/\partial x^i$, where X^i, Y^i are smooth functions on M. If we define $X^i_j := \partial X^i/\partial x^j$ and $Y^i_j := \partial Y^i/\partial x^j$, $1 \leqslant i, j \leqslant n$, then the Lie bracket is locally given by the formula $[X, Y] = \sum_{1\leqslant i,j\leqslant n}(X^j Y^i_j - Y^j X^i_j)\partial/\partial x^i$.

In the rest of the paper, we use Einstein's convention assuming summations over repeated indices. In this context, indices $i, j, k, \dots$ will always be natural numbers between 1 and the dimension of the underlying manifold, which will typically be denoted n.

1.2. Example. The *covariant derivative* $(\Gamma, X, Y) \mapsto \nabla_X Y$ is a natural operator that constructs from a linear connection Γ and vector fields X and Y, a vector field $\nabla_X Y$. In local coordinates,

$$\nabla_X Y = \left(\Gamma^i_{jk} X^j Y^k + X^j Y^i_j\right) \frac{\partial}{\partial x^i}, \tag{1}$$

where Γ^i_{jk} are Christoffel symbols.

Natural operations can be composed into more complicated ones. Examples of 'composed' operations are the *torsion* $T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$ and the *curvature* $R(X, Y)Z := \nabla_{[X,Y]} Z - [\nabla_X, \nabla_Y]Z$ of the linear connection Γ.

1.3. Example. Let X be a vector field and ω a 1-form on M. Denote by $\omega(X) \in C^\infty(M)$ the evaluation of the form ω on X. Then $(X, \omega) \mapsto \exp(\omega(X))$ defines a natural differential operator with values in smooth functions. Clearly, the exponential can be replaced by an arbitrary smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, giving rise to a natural operator $\mathfrak{O}_\varphi(X, \omega) := \varphi(\omega(X))$.

1.4. Example. 'Randomly' generated local formulas need not lead to natural operators. As we will see later, neither $O_1(X, Y) = X^1_3 Y^4 \partial/\partial x^2$ nor $O_2(X, Y) = X^j Y^i_j \partial/\partial x^i$ behaves properly under coordinate changes, so they do not give rise to vector-field valued natural operators.