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Non-zero contact and Sasakian reduction ☆

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Abstract

We complete the reduction of Sasakian manifolds with the non-zero case by showing that Willett's contact reduction is compatible with the Sasakian structure. We then prove the compatibility of the non-zero Sasakian (in particular, contact) reduction with the reduction of the Kähler (in particular, symplectic) cone. We provide examples obtained by toric actions on Sasakian spheres and make some comments concerning the curvature of the quotients.

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1. Introduction

1.1. Sasakian manifolds

We start by briefly recalling the notion of a Sasakian manifold, sending to [4,5] for more details and examples.

Definition 1.1. A *Sasakian* manifold is a $(2n+1)$-dimensional Riemannian manifold (M, g) endowed with a unitary Killing vector field ξ such that the curvature tensor of g satisfies the equation:

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi \tag{1.1}$$

where η is the metric dual 1-form of ξ: $\eta(X) = g(\xi, X)$.

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It can be seen that η is a contact form (with Reeb field ξ). Using the Killing property of ξ and Eq. (1.1), one defines an almost complex structure on the contact distribution $\operatorname{Ker}\eta$, by (the restriction of) $\varphi = \nabla\xi$, where ∇ is the Levi-Civita connection of g.

The following formulae are then easily deduced:

$$\varphi\xi = 0, \qquad g(\varphi Y, \varphi Z) = g(Y, Z) - \eta(Y)\eta(Z). \tag{1.2}$$

The simplest compact example is the round sphere $S^{2n-1} \subset \mathbb{C}^n$, with the metric induced by the flat one of $\mathbb{C}^n$. The characteristic Killing vector field is $\xi_p = -i\vec{p}$, i being the imaginary unit. More general Sasakian structures on the sphere can be obtained by deforming this standard structure as follows. Let $\eta_A = \frac{1}{\sum a_j|z_j|^2}\eta_0$, for $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. Its Reeb field is $R_A = \sum a_j(x_j\partial y_j - y_j\partial x_j)$. Clearly, η_0 and η_A underly the same contact structure. Define the metric g_A by the conditions:

- $g_A(X, Y) = \frac{1}{2}d\eta_A(IX, Y)$ on the contact distribution (here I is the standard complex structure of $\mathbb{C}^n$);
- R_A is normal to the contact distribution and has unit length.

It can be seen that $S_A^{2n-1} := (S^{2n-1}, g_A)$ is a Sasakian manifold (cf. [9]). It has recently been shown in [11] that each compact Sasakian manifold admits a CR-immersion in a S_A^{2N-1}.

Sasakian manifolds, especially the Sasakian–Einstein ones, seem to be more and more important in physical theories (connected with the Maldacena conjecture). Many new examples appeared lately, especially in the work of Ch.P. Boyer, K. Galicki and their collaborators.

This growing importance of Sasakian structures was the first motivation for extending in [8] the contact (zero) reduction to this metric setting, by showing that the contact reduction is compatible with the Sasakian data.

A good procedure for contact reduction away from zero was not available when the paper [8] was written. We here complete the missing picture by showing that Willett's recently defined non-zero reduction introduced in [13] is compatible with the Sasakian data.

1.2. Contact reduction

1.2.1. Contact reduction at 0 *following* [1,7]

Let (M^{2n-1}, η) be an exact contact manifold: this means that η is a contact form ($\eta \wedge (d\eta)^n \neq 0$), hence its kernel is a contact structure on M.

Let R be the Reeb vector field, characterized by the conditions $\eta(R) = 1$ and $d\eta(R, \cdot) = 0$. The flow of the (nowhere vanishing) Reeb vector field preserves the contact form η.

Let $\Phi : G \times M \to M$ be an action by strong contactomorphisms of a (finite dimensional) Lie group on M: for any $f \in G$, $f^*\eta = \eta$.[1] Such a G-action by strong contactomorphisms on (M, η) always admits an equivariant momentum map $J : M \to \mathfrak{g}^*$ given by evaluating the contact form on fundamental fields: $\langle J, \xi\rangle = \eta(\xi_M)$.[2] Note the main difference towards the symplectic case: an action by contactomorphisms is automatically Hamiltonian.

It can be seen that $0 \in \mathfrak{g}^*$ is a regular value for J if and only if the fundamental fields induced by the action do not vanish on the zero level set of J. In this case, the pull back of the contact form to $J^{-1}(0)$ is basic. Let $\pi_0 : J^{-1}(0) \to J^{-1}(0)/G$ and $\iota_0 : J^{-1}(0) \hookrightarrow M$ be the canonical projection (we shall always suppose that the considered actions are free and proper, although these hypothesis can be relaxed to deal with the category of orbifolds) and inclusion respectively. Albert's reduction theorem assures the existence of a unique contact form η_0 on $J^{-1}(0)/G$ such that $\pi_0^*\eta_0 = \iota_0^*\eta$. It can be seen that the contact structure of the quotient depends only on the contact structure on M.

The Sasakian version of this result states (cf. [8]) that if M is Sasakian and G acts by isometric strong contactomorphisms, then the metric also projects to the contact quotient and the whole structure is Sasakian.

[1] If the action is proper or G is compact, this is not more restrictive than asking G to preserve only the contact structure: in the first case, one uses a Palais type argument, in the second case an invariant contact form can be found by averaging.

[2] Here and in the sequel, for a $X \in \mathfrak{g}$, X_M denotes the fundamental field it induces on M.