Available online at www.sciencedirect.com
ScienceDirect
Differential Geometry and its Applications 25 (2007) 251–257
DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
www.elsevier.com/locate/difgeo

On the fundamental group of some open manifolds

Nader Yeganefar

Laboratoire d'Analyse, Topologie, Probabilités, CMI, Université de Provence, 39 rue F. Joliot-Curie, 13453 Marseille cedex 13, France

Received 17 November 2005
Available online 15 December 2006
Communicated by O. Kowalski

Abstract

We study fundamental groups of noncompact Riemannian manifolds. We find conditions which ensure that the fundamental group is trivial, finite or finitely generated.

MSC: 53C21

Keywords: Fundamental group; Volume growth; Diameter growth

1. Introduction

Let M be a complete noncompact Riemannian manifold. It is a classical theme in Riemannian geometry to find geometric conditions which ensure some finiteness results for the topology of M. For example, what conditions would imply that M has finite topological type (i.e. is homeomorphic to the interior of a compact manifold with boundary)? Answers to this question are numerous and can be easily found in the literature, but we would not give any reference here.

In this note, we are merely interested in the fundamental group of M. As M is not compact, the basic problem is to know if its fundamental group is finitely generated or not, and then to know if it is finite or even trivial. Here, we will deal mainly with two situations.

First, we assume that M has sectional curvature K bounded below by a negative constant, i.e. $K \geqslant -1$. Denote by $V_{\mathbb{H}^n}(r)$ the volume of a ball of radius r in hyperbolic space $\mathbb{H}^n$. Then

$$V_{\mathbb{H}^n}(r) = \omega_{n-1} \int_0^r \sinh^{n-1}(t)\, dt,$$

where ω_{n-1} is the volume of the unit sphere $S^{n-1} \subset \mathbb{R}^n$. The Bishop–Gromov theorem [9] asserts that for all $p \in M$, the function $r \mapsto \operatorname{vol}(B(p,r))/V_{\mathbb{H}^n}(r)$ is decreasing, where $B(p,r)$ is a ball of radius r around p in M. (Of course,

E-mail address: yeganefa@cmi.univ-mrs.fr.

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2006.11.004

a lower bound on Ricci curvature is enough for this.) Therefore, the quantity

$$v_p := \lim_{r\to\infty} \frac{\mathrm{vol}\,(B(p,r))}{V_{\mathbb{H}^n}(r)} \in [0,1]$$

is well defined. We also set

$$v(M) = \inf_{p\in M} v_p.$$

Manifolds with $v(M) > 0$ are said to have large volume growth and were studied by Xia [13], who found additional conditions which imply that M has finite topological type or is diffeomorphic to $\mathbb{R}^n$. Here, our first result is

Theorem 1.1. *Let M^n be a complete n-dimensional noncompact Riemannian manifold, and fix a point $p \in M$. Assume that the sectional curvature K is bounded below $K \geqslant -1$. For any real number $L > 0$, there exists a constant $v = v(L,n) \in (0,1)$ such that if*

$$\lim_{r\to\infty} \frac{\mathrm{vol}\,B(p,r)}{V_{\mathbb{H}^n}(r)} \geqslant 1 - v$$

and M is not simply connected, then the length of the shortest homotopically nontrivial geodesic loop based at p is bigger than L.

This may be seen as a noncompact version of a result of Cheeger [4].

For $r > 0$, set

$$\delta(r) = \frac{\sqrt{\cosh(r)}}{\cosh(r/2)}.$$

Combining our theorem with techniques developed by Xia in [13], we get

Corollary 1.2. *Let (M^n, g) be a complete n-dimensional noncompact Riemannian manifold, and fix a point $p \in M$. Assume that the sectional curvature K is bounded below $K \geqslant -1$. Given a real number $L > 0$, there exists $v = v(L,n) \in (0,1)$ such that if*

(i)

$$\lim_{r\to\infty} \frac{\mathrm{vol}\,B(p,r)}{V_{\mathbb{H}^n}(r)} \geqslant 1 - v,$$

(ii) *M has large volume growth, i.e. $v(M) := \inf_{q\in M} v_q > 0$,*

(iii) *and for all $r \geqslant L$,*

$$\frac{\mathrm{vol}\,B(p,2r) - v_p V_{\mathbb{H}^n}(2r)}{v(M)\omega_{n-1}} < \int_0^{\cosh^{-1}(\delta(2r))} \sinh^{n-1}(t)\,dt,$$

then M is simply connected.

Next we consider manifolds without curvature bounds but we impose some growth conditions at infinity. More specifically, if M is any complete manifold and $p \in M$ is any point, we can study the asymptotic behavior of the diameter of geodesic spheres around p, as the radius goes to infinity. Let $\mathrm{diam}\,\partial B(p,r)$ denote the diameter of a sphere of radius r around p, measured with respect to the distance of M. By the triangle inequality, we have always

$$\mathrm{diam}\,\partial B(p,r) \leqslant 2r.$$

For example, if there is a line in M passing through p, then the diameter of a sphere of radius r around p is exactly $2r$. On the other hand, in [11] Z. Shen considered manifolds satisfying

$$\limsup_{r\to\infty} \frac{\mathrm{diam}\,\partial B(p,r)}{r} < 1. \tag{1.1}$$