Available online at www.sciencedirect.com
ScienceDirect
Differential Geometry and its Applications 25 (2007) 277–280
DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS
www.elsevier.com/locate/difgeo

Sussmann's orbit theorem and maps ☆

Benjamin McKay

University College Cork, Cork, Ireland

Received 1 December 2005
Available online 1 December 2006
Communicated by D.V. Alekseevsky

Abstract

A map between manifolds which matches up families of complete vector fields is a fiber bundle mapping on each orbit of those vector fields.

MSC: 93B29; 58A30

Keywords: Orbit theorem; Control theory; Hopf fibration

1. Introduction

Definition 1. Write $e^{tX}(m) \in M$ for the flow of a vector field X through a point m after time t. Let $\mathfrak{F}$ be a family of smooth vector fields on a manifold M. The *orbit* of $\mathfrak{F}$ through a point $m \in M$ is the set of all points $e^{t_1 X_1} e^{t_2 X_2} \dots e^{t_k X_k}(m)$ for any vector fields $X_j \in \mathfrak{F}$ and numbers t_j (positive or negative) for which this is defined.

Example 1. The vector field $\frac{\partial}{\partial \theta}$ on the Euclidean plane (in polar coordinates) has orbits the circles around the origin, and the origin itself.

Example 2. The set of smooth vector fields supported in a disk has as orbits the open disk (a 2-dimensional orbit) and the individual points outside or on the boundary of the disk (zero dimensional orbits).

Example 3. On Euclidean space, the set of vector fields supported inside a ball, together with the radial vector field coming from the center of the ball, forms a set of vector fields with a single orbit.

Example 4. Translation in a generic direction on a flat torus has densely winding orbits.

☆ This work was supported in full or in part by a grant from the University of South Florida St. Petersburg New Investigator Research Grant Fund. This support does not necessarily imply endorsement by the University of research conclusions.
E-mail address: B.McKay@UCC.ie.

0926-2245/$ – see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.difgeo.2006.11.005

Héctor Sussmann [2–4] proved that the orbits of any family of smooth vector fields are immersed submanifolds (also see Stefan [1]). We prove that a mapping between two manifolds which carries one family of complete vector fields into another, is a fiber bundle mapping on each orbit.

2. Proofs

For completeness, we prove Sussmann's theorem.

Theorem 1. (Sussmann [2]) *The orbit of any point under any family of smooth vector fields is an immersed submanifold* (*in a canonical topology*). *If two orbits intersect, then they are equal. Let $\bar{\mathfrak{F}}$ be the largest family of smooth vector fields which have the same orbits as the given family $\mathfrak{F}$. Then $\bar{\mathfrak{F}}$ is a Lie algebra of vector fields, and a module over the algebra of smooth functions.*

Remark 1. Obviously, one could localize these results, replacing globally defined vector fields with subsheaves of the sheaf of locally defined smooth vector fields.

Proof. We can replace $\mathfrak{F}$ by $\bar{\mathfrak{F}}$ without loss of generality. Therefore, if $X, Y \in \mathfrak{F}$, we can suppose that $e_*^X Y \in \mathfrak{F}$ since the flow of $e_*^X Y$ is

$$e^{t(e_*^X Y)} = e^X e^{tY},$$

which must preserve orbits. We refer to this process as *pushing around* vector fields.

Fix attention on a specific orbit. For each point $m_0 \in M$, take as many vector fields as possible $X_1, \dots, X_k$, out of $\mathfrak{F}$, which are linearly independent at m. Refer to the number k of vector fields as the *orbit dimension*. Pushing around convinces us that the orbit dimension is a constant throughout the orbit. Refer to the map

$$(t_1, \dots, t_k) \in \text{ open } \subset \mathbb{R}^k \mapsto e^{t_1 X_1} \dots e^{t_k X_k} m_0 \in M$$

(which we will take to be defined in some open set on which it is an embedding) as a *distinguished chart* and its image as a *distinguished set*. The tangent space to each point $e^{t_1 X_1} \dots e^{t_k X_k} m_0$ of a distinguished set is spanned by the linearly independent vector fields

$$X_1, e_*^{t_1 X_1} X_2, \dots, e^{t_1 X_1} \dots e_*^{t_{k-1} X_{k-1}} X_k,$$

which belong to $\mathfrak{F}$, since they are just pushed around copies of the X_j. Let Ω be a distinguished set. Suppose that $Y \in \mathfrak{F}$ is a vector field, which is not tangent to Ω. Then at some point of Ω, Y is not a multiple of those pushed around vector fields, so the orbit dimension must exceed k.

Therefore all vector fields in $\mathfrak{F}$ are tangent to all distinguished sets. So any point inside any distinguished set stays inside that set under the flow of any vector field in $\mathfrak{F}$, at least for a short time. So such a point must also stay inside the distinguished set under compositions of flows of the vector fields, at least for short time. Therefore a point belonging to two distinguished sets must remain in both of them under the flows that draw out either of them, at least for short times. Therefore that point belongs to a smaller distinguished set lying inside both of them. Therefore the intersection of distinguished sets is a distinguished set.

We define an open set of an orbit to be any union of distinguished sets; so the orbit is locally homeomorphic to Euclidean space. We can pick a countable collection of distinguished sets as a basis for the topology. Every open subset of M intersects every distinguished set in a distinguished set, so intersects every open set of the orbit in an open set of the orbit. Thus the inclusion mapping of the orbit into M is continuous. Since M is metrizable, the orbit is also metrizable, so a submanifold of M. The distinguished charts give the orbit a smooth structure. They are smoothly mapped into M, ensuring that the inclusion is a smooth map. □

Example 5. Let $\alpha = dy - z\,dx$ in $\mathbb{R}^3$. The vector fields on which $\alpha = 0$ have one orbit: all of $\mathbb{R}^3$, since they include $\partial_z, \partial_x + z\partial_y$, and therefore include the bracket:

$$[\partial_z, \partial_x + z\partial_y] = \partial_y.$$