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Harmonic sections of Riemannian vector bundles, and metrics of Cheeger–Gromoll type ☆

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Abstract

We study harmonic sections of a Riemannian vector bundle $\mathcal{E} \to M$ when $\mathcal{E}$ is equipped with a 2-parameter family of metrics $h_{p,q}$ which includes both the Sasaki and Cheeger–Gromoll metrics. For every $k > 0$ there exists a unique p such that the harmonic sections of the radius-k sphere subbundle are harmonic sections of $\mathcal{E}$ with respect to $h_{p,q}$ for all q. In both compact and non-compact cases, Bernstein regions of the (p,q)-plane are identified, where the only harmonic sections of $\mathcal{E}$ with respect to $h_{p,q}$ are parallel. Examples are constructed of vector fields which are harmonic sections of $\mathcal{E} = TM$ in the case where M is compact and has non-zero Euler characteristic.

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1. Introduction

The aim of this paper is to introduce new criteria for deciding which smooth vector fields on a smooth, connected (not necessarily compact) Riemannian manifold (M, g), or more generally which smooth sections σ of a smooth Riemannian vector bundle $(\mathcal{E}, \langle , \rangle, \nabla) \to M$, qualify as "better than the rest". In so doing we overcome some limitations of existing criteria, two of which we briefly review.

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(1) *σ is a harmonic section of $\mathcal{E}$* [13,14]. Here one measures the *vertical energy* (or *total bending* [20]) of σ:

$$E^v(\sigma) = \frac{1}{2}\int_M |\nabla\sigma|^2 \operatorname{vol}(g), \tag{1.1}$$

(assuming for convenience that M is compact; otherwise one works over relatively compact domains), and looks for critical points with respect to smooth variations through sections of $\mathcal{E}$. The Euler–Lagrange equations are linear:

$$\nabla^*\nabla\sigma = 0, \tag{1.2}$$

where $\nabla^*\nabla$ is the *rough Laplacian*:

$$\nabla^*\nabla = -\operatorname{Trace}\nabla^2.$$

If M is compact then integration by parts shows that all harmonic sections of $\mathcal{E}$ are parallel; therefore if the Euler class $\chi(\mathcal{E}) \neq 0$ there are no non-trivial solutions. The same is true if M is non-compact, provided $|\sigma|^2$ is a harmonic function (Lemma 3.4).

(2) *$|\sigma| = k$, and σ is a harmonic section of the radius-k sphere bundle* [22,24]. Here the functional (1.1) is restricted to sections of $S\mathcal{E}(k) \to M$, where:

$$S\mathcal{E}(k) = \{e \in \mathcal{E}\colon\ |e| = k\},$$

and this constraint causes the Euler–Lagrange equations to become non-linear:

$$\nabla^*\nabla\sigma = \frac{1}{k^2}|\nabla\sigma|^2\sigma. \tag{1.3}$$

Solutions of (1.3) clearly include all parallel sections of length k (if any), but when $\mathcal{E} = TM$ many additional solutions have been identified [1,8,9,17,19], and examined for stability [2–4,12,24]. However the theory is limited to bundles with $\chi(\mathcal{E}) = 0$.

Our new criteria remove the topological restriction $\chi(\mathcal{E}) = 0$, whilst retaining all solutions of the constrained variational problem (2). The idea is to obtain interesting non-linear equations, such as (1.3), by altering the background metric data, rather than introducing constraints. Note first that (1.1) is equivalent to:

$$E^v(\sigma) = \frac{1}{2}\int_M |d^v\sigma|^2 \operatorname{vol}(g), \tag{1.4}$$

where $d^v\sigma$ is the vertical component of the differential $d\sigma$ with respect to ∇, and the norm in $T\mathcal{E}$ is that of the *Sasaki metric h* on $\mathcal{E}$ [18]. We study the functional (1.4) when h is generalized to a 2-parameter family of metrics $h_{p,q}$ on $\mathcal{E}$, for which $h_{0,0} = h$ and $h_{1,1}$ is the *Cheeger–Gromoll metric* [7,15]. (Both the Sasaki and Cheeger–Gromoll metrics generalize in a natural way to vector bundles.) Other geometrically interesting metrics occur in this family; for example $h_{2,0}$ is the stereographic metric.

The term "metric" is used somewhat informally. If $q \geqslant 0$ then $h_{p,q}$ is indeed a Riemannian metric. However if $q < 0$ then $h_{p,q}$ has varying signature: it is Riemannian within a ball bundle of radius $1/\sqrt{-q}$, Lorentzian on the exterior, and positive semi-definite on the boundary. This behaviour is a manifestation of *Kato's inequality* [6]. A section whose image lies in the closure of this ball bundle is said to be *q-Riemannian.* In general, if σ is stationary for (1.4) with respect to $h_{p,q}$ and smooth variations through sections of $\mathcal{E}$ we say that σ is a *(p,q)-harmonic section of $\mathcal{E}$*. The Euler–Lagrange equations are derived in Section 3 (Theorem 3.6). In general they are nonlinear. However the parallel sections of $\mathcal{E}$ are (p,q)-harmonic for all (p,q).

The $h_{p,q}$ induce a vertically homothetic family of Riemannian metrics on $S\mathcal{E}(k)$, even when $q < 0$ and $k > 1/\sqrt{-q}$. Thus (p,q)-harmonic sections of $S\mathcal{E}(k)$ are characterized by Eq. (1.3) for all (p,q), and may therefore be referred to simply as *harmonic sections of $S\mathcal{E}(k)$*. For bundles with $\chi(\mathcal{E}) = 0$ we establish the following:

Theorem A. *Suppose that σ has constant length $k > 0$. Then σ is a (p,q)-harmonic section of $\mathcal{E}$ if and only if σ is parallel, except when $p = 1 + 1/k^2$ in which case σ is a (p,q)-harmonic section of $\mathcal{E}$ if and only if σ is a harmonic section of $S\mathcal{E}(k)$.*