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A remark on renormalized volume and Euler characteristic for ACHE 4-manifolds

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Abstract

This note computes a renormalized volume and a renormalized Gauss–Bonnet–Chern formula for asymptotically complex hyperbolic Einstein (so-called ACHE) 4-manifolds.

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1. Introduction

Asymptotically symmetric Einstein metrics exhibit many interesting phenomena [4,13]. They were especially studied in the Einstein asymptotically hyperbolic (or AHE) case, which enjoys fruitful relationships with physics through the ADS-CFT correspondence, but also is a useful tool establishing links between the conformal geometry of a compact $(n-1)$-dimensional manifold (usually called the *boundary at infinity*) and the Riemannian geometry of a complete Einstein n-dimensional manifold (the bulk AHE manifold). In this setting, an intriguing invariant, called *renormalized volume*, has been defined by C.R. Graham [12], after works by physicists such as Henningson and Skenderis [14].

In even dimensions n, the renormalized volume is an invariant of the AHE metric. Its role in the formula for the Euler characteristic of the Einstein manifold has been moreover pointed out by M.T. Anderson [2] in dimension 4, by S.-Y.A. Chang, J. Qing and P. Yang [7] and by P. Albin [1] in higher dimensions, with applications in dimension 4 to the study of the moduli space of Einstein asymptotically real hyperbolic metrics [3]. This is a "renormalized Gauss–Bonnet–Chern formula" since the Einstein manifold is non-compact, but all divergent terms in the integrals of the formula are shown to cancel, whereas renormalized volume appears as a finite limit contribution.

In odd dimensions n, the renormalized volume is not an invariant of the AHE metric only but depends on a choice of a representative metric on the boundary at infinity in its conformal class. This makes it no less interesting, as it

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gives rise to the so-called *conformal anomaly* phenomenon: the difference between the renormalized volumes of two different choices of metric singles out a local differential operator on the boundary with nice properties [12].

The goal of this short note is to point out analogous results in the case of Einstein asymptotically *complex* hyperbolic (or ACHE) manifolds of dimension 4, where the boundary at infinity is now a strictly pseudoconvex CR 3-manifold, with the hope that such an object would be interesting for the study of 3-dimensional CR geometry.

Starting with an Einstein asymptotically complex hyperbolic (ACHE) metric g on a 4-dimensional M with boundary at infinity a CR manifold X, it has been shown in [5] that there exists a (formal) Kähler–Einstein metric $\bar{g}$ on a neighbourhood of infinity on M whose first terms are formally determined by the choice of a compatible contact form (or in the usual language of CR geometry: a *pseudo-hermitian structure*) on X. One can then show that it is always possible to find a diffeomorphism ψ inducing the identity at infinity so that $\psi^* g$ admits a nice asymptotic development whose first terms are given by the metric $\bar{g}$.

The volume form of $\psi^* g$ itself has an asymptotic development, and we then define the *renormalized volume* as the term of order 0 in this expansion. This is well defined, as we can prove:

Theorem 1.1. *Let (M, g) be a 4-dimensional Einstein asymptotically complex hyperbolic (ACHE) manifold, with boundary at infinity a compact strictly pseudoconvex CR 3-dimensional manifold X. For any choice of compatible contact form η on X, let $\bar{g}$ be the associated formal Kähler–Einstein metric in a neighbourhood of infinity in M, and ψ a diffeomorphism as above. Then the renormalized volume V is well defined and only depends on g and the choice of η on X; we shall call it the* renormalized volume *of (M, g)* relative to η.

A detailed analysis of the boundary term of the expression of the Euler characteristic of M in terms of the curvature of g shows that a *renormalized Gauss–Bonnet characteristic formula* can be obtained. The result reads as follows:

Theorem 1.2. *Let (M, g) be a 4-dimensional Einstein asymptotically complex hyperbolic (ACHE) manifold, with boundary at infinity a compact strictly pseudoconvex CR 3-dimensional manifold X. For any choice of compatible contact form η leading to a renormalized volume V, one has*

$$\chi(M) - \frac{1}{8\pi^2}\int_M |W^g|^2 - \frac{(\mathrm{Scal}^g)^2}{24} = \frac{3}{8\pi^2}V - \frac{1}{4\pi^2}\int_X \left(\frac{R^2}{16} - \frac{5}{2}|\tau|^2\right)\eta\wedge d\eta, \tag{1.1}$$

where R and τ are the curvature and torsion of the Webster–Tanaka connection determined by η.

Note that it is shown in [5] that the integrals of both terms

$$|W^-|^2 \quad \text{and} \quad |W^+|^2 - \frac{1}{24}\mathrm{Scal}^2$$

converge on an ACHE manifold, so that the formula above makes sense.

By applying the previous theorem, we get another proof of the fact that the renormalized volume depends only on (M, g) and η, since it explicitly relates V with objects defined only in terms of g and a choice of contact form at infinity. Moreover, this implies of course that the number

$$\mathcal{V} = \frac{3}{2}V - \int_X \left(\frac{R^2}{16} - \frac{5}{2}|\tau|^2\right)\eta\wedge d\eta \tag{1.2}$$

is an invariant of the ACHE manifold (M, g) only.

The situation depicted in Theorem 1.2 is then less pleasant than in the AHE case, as the renormalized volume is never an invariant of the complete Einstein metric and always depends on the choice of a contact form at infinity. As the model case of the complex hyperbolic plane shows, the appearance of a local correction on the boundary seems unavoidable.

This situation is of course reminiscent from the AHE case of odd dimension (i.e. boundary at infinity of *even* dimension); this should come as no surprise as it is usually considered that CR geometry enjoys lots of analogies with even-dimensional conformal geometry. It moreover shows that, rather than giving rise to a global invariant, the renormalized volume gives birth to a conformal (or rather CR) *anomaly*, i.e. a formula relating the renormalized