Available online at www.sciencedirect.com

ScienceDirect

Journal of Approximation Theory 213 (2017) 23–49

JOURNAL OF Approximation Theory

www.elsevier.com/locate/jat

Full length article

Democracy of shearlet frames with applications

Daniel Vera*

Departamento de Matemáticas, ITAM, Río Hondo No. 1, Ciudad de México 01080, Mexico

Received 15 February 2016; received in revised form 3 October 2016; accepted 21 October 2016
Available online 2 November 2016

Communicated by Gitta Kutyniok

Abstract

Shearlets on the cone provide Parseval frames for L^2. They also provide near-optimal approximation for the class $\mathcal{E}$ of cartoon-like images. Moreover, there are spaces associated to them other than L^2 and there exist embeddings between these and classical spaces.

We prove approximation properties of the cone-adapted shearlet system coefficients in a more general context. Namely, when the target shearlet sequence belongs to a class or space different to that obtained from a shearlet sequence of a $f \in \mathcal{E}$ and when the error is not necessarily measured in the L^2-norm (or, since the shearlet system is a frame, the ℓ^2-norm) but in a norm of a much wider family of smoothness spaces of "high" anisotropy. We first prove democracy of shearlet frames in shear anisotropic inhomogeneous Besov and Triebel–Lizorkin sequence spaces. Then, we prove embeddings between approximation spaces and discrete weighted Lorentz spaces in the framework of shearlet coefficients. Simultaneously, we also prove that these embeddings are equivalent to Jackson and Bernstein type inequalities. This allows us to find real interpolation between these highly anisotropic sequence spaces. We also describe how some of these results can be extended to other shearlet and curvelet generated spaces. Finally, we show some examples of embeddings between wavelet approximation spaces and shearlet approximation spaces and obtain a similar result stated in $L^2(\mathbb{R}^2)$ for the curvelet smoothness spaces. This also paves the way to the use of thresholding algorithms in compression or noise reduction.

MSC: 41A05; 41A17; 42B35; 42C15; 42C40

Keywords: Approximation spaces; Democracy; Frames; Parabolic molecules; Shearlets; Smoothness spaces

* Fax: +52 (55) 5628 4086., +52 5556284086.
E-mail address: daniel.vera@itam.mx.

http://dx.doi.org/10.1016/j.jat.2016.10.003
0021-9045/

1. Introduction

The framework of wavelets provides orthonormal bases for L^2 and, more generally, frames that have been successfully applied in harmonic analysis, for the numerical and analytical solution of certain partial differential equations as well as in signal processing and statistical estimation. Not only can wavelets be used to characterize some classical spaces but there are also implementable fast algorithms. Although wavelets provide better approximation properties than Fourier techniques, they lack of high directional sensitivity in dimensions $d \geq 2$ since the number of wavelets (from a multi resolution analysis) remains constant across scales: $2^d - 1$. Some directional systems with increasing anisotropy (following a parabolic scaling law) have been created to overcome this limitation. Two of them are the *curvelets* of Candès and Donoho [2,3] and the *shearlets* on the cone of Guo, Kutyniok and Labate [17].

1.1. Approximation and wavelets

Approximation theory benefits from the unconditional bases provided by wavelet systems since "it is enough to threshold the properly normalized wavelet coefficients" (see [8]) to achieve good N-term nonlinear approximation. It is also well known that the approximation order is closely related to the smoothness of the function. A generalization of the nonlinear approximation theory, called restricted nonlinear approximation (RNLA), was carried out by Cohen, DeVore and Hochmuth in [5] in the setting of wavelet bases in Hardy and Besov spaces where the authors control the measure ν of the index set $\mathcal{D}$ of the approximation elements instead of the number of terms in the approximation. This measure (generally other than the counting measure) is closely related to a weighting of the coefficients in the wavelet expansion. A brief discussion on this weight sequence will be given at the end of the present paragraph. One of the novelties in [5] is that the approximation spaces are not necessarily contained in the space in which the error is measured. Some extensions of the restricted nonlinear approximation were done in [24,22] for general bases in quasi-Banach distribution spaces and quasi-Banach lattice sequence spaces, respectively. We develop our results based on the restricted non-linear approximation in sequence spaces (RNLASS) theory in [22] since its framework is more general than [24] (in fact, [22] allows to recover results in [5], in contrast to [24]). Approximation theory is related to real interpolation in a way that some identifications between approximation spaces, interpolations spaces, discrete Lorentz spaces and Besov spaces can be proved for certain parameters. These identifications turn out to be equivalent to what is known as democracy and to Jackson and Bernstein type inequalities (see [5,24,22] and references therein). The concept of democracy (of a basis or a frame in a space) is related to that of p-space which, in turn, is better known as p-Temlyakov property (see Section 2.1 for definitions): If, for $C > 0$, the condition

$$\frac{1}{C}(\nu(\Gamma))^{1/p} \leq \left\| \sum_{I\in\Gamma} \frac{\mathbf{e}_I}{u_I} \right\|_{\mathfrak{f}} \leq C(\nu(\Gamma))^{1/p}, \tag{1.1}$$

holds for all $\Gamma \subset \mathcal{D}$ such that $\nu(\Gamma) < \infty$ we say that $(\mathfrak{f}, \mathcal{B}, \nu)$ shares the p-Temlyakov property, where $\mathfrak{f}$ is a quasi-Banach space, $\mathcal{B} = \{\mathbf{e}_I\}_{I\in\mathcal{D}}$ is an unconditional basis for $\mathfrak{f}$, $\mathbf{u} = \{u_I\}_{I\in\mathcal{D}}$ is a weight sequence with $u_I > 0$ for all $I \in \mathcal{D}$ and ν is a measure on the index set. The measure ν and the sequence $\mathbf{u}$ are intimately related. Usually, the thresholding approximation error is measured in the same space as that of the signal representation. It is argued in [23] that procedures, as thresholding approximants, having small error in L^p, $p \neq 2$, may reflect better the visual properties of a signal. In [5] this situation is generalized: The wavelet coefficients