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Banach–Steinhaus theory revisited: Lineability and spaceability

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Abstract

In this paper we study the divergence behavior of linear approximation processes in general Banach spaces. We are interested in the structure of the set of vectors creating divergence. The Banach–Steinhaus theory gives some information about this set, however, it cannot be used to answer the question whether this set contains subspaces with linear structure. We give necessary and sufficient conditions for the lineability and the spaceability of the set of vectors creating divergence.

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1. Introduction and notation

A central problem in approximation theory is the approximation of a bounded linear operator T by a sequence of operators $\{T_N\}_{N\in\mathbb{N}}$. We consider the following setting. Let B_1 and B_2 be two Banach spaces and let $T: B_1 \to B_2$ be a bounded linear operator. Given a sequence of bounded

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linear operators $\{T_N\}_{N\in\mathbb{N}}$ mapping B_1 into B_2, we are interested in whether, for all $f \in B_1$, $T_N f$ converges to Tf in the norm of B_2. According to the Banach–Steinhaus theorem, the answer to this question is "yes" if and only if there exists a constant C such that $\|T_N\|_{B_1\to B_2} \le C$ for all $N \in \mathbb{N}$ and we have $T_N f \to Tf$ as N tends to infinity for all f from a dense subspace of B_1.

On the other hand, the Banach–Steinhaus theorem also implies the principle of condensation of singularities: If there exists a vector $f \in B_1$ for which we have divergence, i.e., $\limsup_{N\to\infty} \|T_N f\|_{B_2} = \infty$, then we have divergence for all vectors from a residual and therefore dense subset of B_1.

Since the publication of Banach and Steinhaus [5,4], the Banach–Steinhaus theory has been developed further and has today become an important part of functional analysis. There also have been efforts to extend the Banach–Steinhaus theory into different directions [20,12,13,21].

Although the Banach–Steinhaus theorem gives some information about the size of the set of vectors creating divergence, it gives no information about the structure of this set. In particular, it would be interesting to know if it possesses a linear structure. Such a linear structure is important in application, because it implies that any linear combination of vectors, which is not the zero vector, leads to divergence as well.

Note that it is significantly more difficult to show a linear structure in the set of vectors with divergent approximation process compared to showing a linear structure in the set of vectors with convergent approximation process. If we have two vectors f_1 and f_2, for which $T_N f_1$ and $T_N f_2$ converge, it is clear that for their sum $f_1 + f_2$ we have convergence as well. Hence, any finite linear combination of vectors with convergent approximation process will be a vector with the same property. However, for divergence this is not true. Given two vectors g_1 and g_2 for which $T_N g_1$ and $T_N g_2$ diverge, we cannot conclude that $T_N(g_1 + g_2)$ diverges: indeed, choose $g_2 = f_1 - g_1$, where f_1 is any vector with convergent approximation process and g_1 any vector with divergent approximation process.

The above example shows that in general we cannot expect that the set of vectors with divergent approximation process is a linear space. However, we can ask if this set contains an infinite dimensional subspace with linear structure.

Lineability and spaceability are two mathematical concepts that are suitable to study this question. They were recently introduced and used for example in [15,16,3,6,1] (see also the recent book [2]). Both describe the structure of some given subset of an ambient normed space or, more generally, linear topological space. A set S in a linear (linear topological, resp.) space X is said to be lineable (spaceable, resp.) if $S \cup \{0\}$ contains an infinite dimensional (a closed infinite dimensional, resp.) subspace of X. Results about lineability and spaceability have been obtained for different problems, see [2]. As an example concerning divergence, Bayart showed in [6] that the set of functions in $L^1(\partial\mathbb{D})$ whose Fourier series diverges everywhere on $\partial\mathbb{D}$ is spaceable. Moreover, he proved in [7] a number of results concerning lineability of families of functions for which the action of certain sequences of continuous linear forms (functionals) on a Banach space generates divergence. Here we adopt a more general point of view by considering sequences of continuous linear mappings between two Banach spaces.

2. General setting

In this paper we present results about the lineability and spaceability of certain sets. In order to be able to discuss the problems, we introduce some notation, which will be used throughout the rest of this paper.