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Comparing the degrees of unconstrained and shape preserving approximation by polynomials

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Abstract

Let $f \in C[-1,1]$ and denote by $E_n(f)$ its degree of approximation by algebraic polynomials of degree $< n$. Assume that f changes its monotonicity, respectively, its convexity finitely many times, say $s \geq 2$ times, in $(-1,1)$ and we know that for $q = 1$ or $q = 2$ and some $1 < \alpha \leq 2$, such that $q\alpha \neq 4$, we have

$$E_n(f) \leq n^{-q\alpha}, \quad n \geq s + q + 1.$$

The purpose of this paper is to prove that the degree of comonotone, respectively, coconvex approximation, of f, by algebraic polynomials of degree $< n$, $n \geq N$, is also $\leq c(\alpha, s)n^{-q\alpha}$, where the constant N depends only on the location of the extrema, respectively, inflection points in $(-1,1)$ and on α.

This answers, affirmatively, questions left open by the authors in papers with Kopotun (in Ukrainian Math. J.) and with Vlasiuk (see the list of references).

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1. Introduction and main results

Let $C[a, b]$, $-1 \le a < b \le 1$, denote the space of continuous functions on $[a, b]$ equipped with the usual uniform norm, $\|f\|_{[a,b]} := \max_{a\le x\le b} |f(x)|$. When dealing with $[-1, 1]$, we suppress referring to the interval, namely, we denote $\|f\| := \|f\|_{[-1,1]}$. For $\mathbb{P}_n$, the space of algebraic polynomials of degree $< n$ and $f \in C[-1, 1]$, denote by

$$E_n(f) := \inf_{p_n\in\mathbb{P}_n} \|f - p_n\|,$$

the degree of approximation of f by algebraic polynomials of degree $< n$.

Given $s \ge 1$, denote by $\mathbb{Y}_s$, the set of all collections $Y_s = \{y_i\}_{i=1}^s$, of points y_i, such that $y_{s+1} := -1 < y_s < \cdots < y_1 < 1 =: y_0$. For such a collection we write $f \in \Delta^{(1)}(Y_s)$ if $f \in C[-1, 1]$ and $(-1)^i f$ is nondecreasing on $[y_{i+1}, y_i]$, $0 \le i \le s$. Similarly, we write $f \in \Delta^{(2)}(Y_s)$ if $f \in C[-1, 1]$ and $(-1)^i f$ is convex on $[y_{i+1}, y_i]$, $0 \le i \le s$.

For $f \in \Delta^{(q)}(Y_s)$, $q \in \{1, 2\}$, we denote by

$$E_n^{(q)}(f, Y_s) := \inf_{P_n\in\mathbb{P}_n\cap\Delta^{(q)}(Y_s)} \|f - P_n\|,$$

the degree of best comonotone, respectively, coconvex approximation of f relative to Y_s.

Assuming that for some $\alpha > 0$ and $N \ge 1$,

$$n^\alpha E_n(f) \le 1, \quad n \ge N, \tag{1.1}$$

the answer to the following question was provided (see [3–5,9]).

If (1.1) holds for an $f \in \Delta^{(q)}(Y_s)$, is it possible to have constants $c(q, \alpha, s, N)$ and N^* such that

$$n^\alpha E_n^{(q)}(f, Y_s) \le c(q, \alpha, s, N), \quad n \ge N^*? \tag{1.2}$$

Here N^*, if it exists, may depend on q, α, s and N, but may also depend on Y_s or even on f. It turns out that N^* always exists and its dependence on the various parameters, in all cases, but $1 < \alpha \le 2$, $N = s + 2$, $s \ge 2$, for the comonotone case ($q = 1$), was given in [5,9] and, in all cases, but $2 < \alpha \le 4$, $N = s + 3$, $s \ge 3$, for the coconvex case ($q = 2$), was given in [3,4].

O.V. Vlasiuk [10], has attempted to close the above gaps, but, regrettably, the proof of the main lemma there is incorrect (see [11]). Our main results are the following.

Theorem 1.1. *Given $Y_s \in \mathbb{Y}_s$, $s \ge 2$, and $1 < \alpha \le 2$. Then, there exist constants $c(\alpha, s)$ and $N^*(\alpha, Y_s)$, such that for all functions $f \in \Delta^{(1)}(Y_s)$ satisfying (1.1) with $N = s + 2$, (1.2) with $q = 1$, holds.*

Theorem 1.2. *Given $Y_s \in \mathbb{Y}_s$, $s \ge 3$, and $2 < \alpha < 4$. Then, there exist constants $c(\alpha, s)$ and $N^*(\alpha, Y_s)$, such that for all functions $f \in \Delta^{(2)}(Y_s)$ satisfying (1.1) with $N = s + 3$, (1.2) with $q = 2$, holds.*

Remark 1.3. Note that this leaves open what happens in the coconvex case when $\alpha = 4$, $N = s + 3 > 5$.

In Section 2 we bring some auxiliary lemmas and in Section 3 we prove Theorems 1.1 and 1.2. Throughout the paper, k, r, s, q, i, j and n, are nonnegative integers, while α, a, b, h, t, u and v, are real numbers.