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On multivariate discrete least squares

Yeon Ju Lee[a,*], Charles A. Micchelli[b], Jungho Yoon[c]

[a] Department of Mathematics, Korea University Sejong Campus, Sejong, 339-700, South Korea
[b] Department of Mathematics and Statistics, SUNY Albany, Albany, 12222, USA
[c] Department of Mathematics, Ewha Womans University, Seoul, 120-750, South Korea

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Abstract

For a positive integer $n \in \mathbb{N}$ we introduce the index set $\mathbb{N}_n := \{1, 2, \ldots, n\}$. Let $X := \{x_i : i \in \mathbb{N}_n\}$ be a *distinct* set of vectors in $\mathbb{R}^d$, $Y := \{y_i : i \in \mathbb{N}_n\}$ a *prescribed* data set of real numbers in $\mathbb{R}$ and $\mathcal{F} := \{f_j : j \in \mathbb{N}_m\}, m < n$, a given set of *real valued continuous* functions defined on some neighborhood $\mathcal{O}$ of $\mathbb{R}^d$ containing X. The discrete least squares problem determines a (generally unique) function $f = \sum_{j \in \mathbb{N}_m} c_j^\star f_j \in \operatorname{span}\mathcal{F}$ which minimizes the square of the ℓ^2-norm

$$\sum_{i \in \mathbb{N}_n} \left(\sum_{j \in \mathbb{N}_m} c_j f_j(x_i) - y_i \right)^2$$

over all vectors $(c_j : j \in \mathbb{N}_m) \in \mathbb{R}^m$. The value of f at some $s \in \mathcal{O}$ may be viewed as the optimally predicted value (in the ℓ^2-sense) of *all* functions in $\operatorname{span}\mathcal{F}$ from the given data $X = \{x_i : i \in \mathbb{N}_n\}$ and $Y = \{y_i : i \in \mathbb{N}_n\}$.

We ask "What happens if the components of X and s are nearly the same". For example, when all these vectors are near the origin in $\mathbb{R}^d$. From a practical point of view this problem comes up in image analysis when we wish to obtain a new pixel value from nearby available pixel values as was done in [2], for a specified set of functions $\mathcal{F}$.

* Corresponding author.
E-mail address: leeyeonju08@korea.ac.kr (Y.J. Lee).

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This problem was satisfactorily solved in the univariate case in Section 6 of Lee and Micchelli (2013). Here, we treat the significantly more difficult multivariate case using an approach recently provided in Yeon Ju Lee, Charles A. Micchelli and Jungho Yoon (2015).

Keywords: Collocation matrix; Wronskian; Multivariate least squares; Multivariate Maclaurin expansion

1. Background and problem formulation

Our goal in this paper is to extend the treatment of *sensitivity analysis* of univariate discrete least squares problems, as in presented in Section 6 of [3], to the *multivariate case*. In that paper, the method of analysis involved a careful use of the Cauchy Binet formula as described in the beginning of the book [1]. Here, we find the use of a finite Maclaurin expansion, as presented recently in [4] most appropriate. This method, joined with some matrix theoretic considerations, allows us to achieve our goal of extending the sensitivity analysis of the univariate discrete least squares presented in Section 6 of [3] to the multivariate case. The contribution of this paper is to analyze some aspects of the multivariate least squares problem discussed only in the univariate case in Section 6 of [3]. In this paper, we treat the multivariate case which is not done in [3]. Nevertheless, we shall, as often as possible, use the notation and setup described in [3] and, when necessary, modify it to follow the discussion in [4].

To start, we let $\mathbb{N}_n = \{1, 2, \ldots, n\}$ where n is a positive integer in $\mathbb{N}$, the set of natural numbers and use $\mathbb{Z}_+^d$ for the set of all nonnegative *lattice vectors* in $\mathbb{R}^d$. That is, $\alpha = (\alpha_j : j \in \mathbb{N}_d) \in \mathbb{Z}_+^d$ means α_j, for all $j \in \mathbb{N}_d$, is a nonnegative integer. Notationally, we write that $\alpha_j \in \mathbb{Z}_+$. Likewise, we set $|\alpha|_1 := \sum_{j\in\mathbb{N}_d} \alpha_j$ and $\alpha! := \alpha_1! \cdots \alpha_d!$. Next, we introduce a *partial ordering* for lattice points in $\mathbb{Z}_+^d$. When $\alpha \neq \beta$, we say $\alpha \prec \beta$, that is, α *comes before* β, provided either $|\alpha|_1 < |\beta|_1$, or $|\alpha|_1 = |\beta|_1$ and there is integer $m \in \mathbb{N}_{d-1}$ such that $\alpha_j = \beta_j$, $j \in \mathbb{N}_m$ and while $\alpha_{m+1} < \beta_{m+1}$. Moreover, we use $\alpha \preceq \beta$ to mean that either $\alpha \prec \beta$ or $\alpha = \beta$. We let $\alpha^{[m]}$ be the mth lattice vector in $\mathbb{Z}_+^d$ relative to our partial ordering and use the symbol $\mathbb{B}_m$ for the set of all lattice vectors below or equal to $\alpha^{[m]}$ in the ordering "$\preceq$", that is,

$$\mathbb{B}_m := \{\alpha : \alpha \in \mathbb{Z}_+^d, \alpha \preceq \alpha^{[m]}\}.$$

We are now ready to describe the multivariate discrete least squares problem which we consider in this paper. We specify a *data set* of n real numbers $Y = \{y_\alpha : \alpha \in \mathbb{B}_n\} \subset \mathbb{R}$ and *data locations* in $\mathbb{R}^d$ given by the set $X = \{x_\alpha : \alpha \in \mathbb{B}_n\}$. That is, the scalar data value y_α corresponds to the data vector x_α for any $\alpha \in \mathbb{B}_n$. The set X determines the vector $\mathbf{x} = (x_\alpha : \alpha \in \mathbb{B}_n) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, ($n$ products of copies of $\mathbb{R}^d$). For simplicity, we merely use $\mathbb{R}^{nd}$ for this product set. In addition, we have m real valued continuous function $\mathcal{F} = \{f_\beta : \beta \in \mathbb{B}_m\}$ defined on an open neighborhood $\mathcal{O}$ of the origin in $\mathbb{R}^d$ where m is assumed to be strictly smaller than n.

Our objective is to represent the data set Y everywhere as a function $f \in \operatorname{span}\mathcal{F}$ which deviates least at x_α from y_α, $\alpha \in \mathbb{B}_n$, in the discrete least squares sense. Thus, if a typical element in $\operatorname{span}\mathcal{F}$ has the form

$$f = \sum_{\beta\in\mathbb{B}_m} c_\beta f_\beta$$