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Strong continuity on Hardy spaces

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Abstract

We prove the strong continuity of spectral multiplier operators associated with dilations of certain functions on the general Hardy space H^1_L introduced by Hofmann, Lu, Mitrea, Mitrea, Yan. Our results include the heat and Poisson semigroups as well as the group of imaginary powers.

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1. Introduction

In the theory of semigroups of linear operators on Banach spaces the crucial assumption is that of strong continuity. One often encounters a situation where the semigroup $T_t = e^{-tL}$ is initially defined on $L^2(\Omega)$ and L is a non-negative self-adjoint operator. In this case the spectral theorem immediately gives the strong $L^2(\Omega)$ continuity $\lim_{t\to 0^+} \|T_t f - f\|_{L^2(\Omega)} = 0$, for $f \in L^2(\Omega)$. Assume additionally that $\{T_t\}_{t>0}$ extends to a locally bounded semigroup on L^p. More precisely, we impose that for each $1 \leq p < \infty$ there exists $t_p > 0$ such that

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$\|T_t\|_{L^p(\Omega)\to L^p(\Omega)} \le C_p, t \in [0, t_p]$. Since weak and strong convergence coincide for semigroups of operators (see e.g. [6, Theorem 5.8]), it is straightforward to see that T_t is strongly continuous on all $L^p(\Omega)$, $1 < p < \infty$. Moreover, if we assume that $\{T_t\}_{t>0}$ is contractive on $L^1(\Omega)$, then it is also strongly continuous on $L^1(\Omega)$. Quite often the semigroup $\{T_t\}_{t>0}$ may be also defined on function spaces other than L^p. For instance, if $T_t = e^{t\Delta}$ is the classical heat semigroup on $\mathbb{R}^d$, then it also acts on the atomic Hardy spaces H^1_{at}. However, even in this case it is not obvious that the semigroup is strongly continuous on H^1_{at}.

In this paper we impose that $\{T_t\}_{t>0}$ satisfies the so-called Davies–Gaffney estimates (see (2.3)), and that the underlying space Ω is a space of homogeneous type in the sense of Coifman–Weiss [1]. Under these assumptions, as a corollary of our main result, we prove that e^{-tL} and $e^{-t\sqrt{L}}$ are strongly continuous on the Hardy space H^1_L. This Hardy space was introduced by Hofmann, Lu, Mitrea, Mitrea, Yan in [8]. Our results are quite general, as there are many operators L satisfying (2.3), e.g. Laplace–Beltrami operators on complete Riemannian manifolds (see e.g. [7, Corollary 12.4]) or Schrödinger operators with non-negative potentials.

The literature on L^p spectral multipliers for operators satisfying Davies–Gaffney estimates is vast. However, as the L^p theory is not discussed in our paper, we do not provide detailed references on this subject. Instead we kindly refer the interested reader to consult e.g. [11] and references therein. There are also results for spectral multipliers on the Hardy space H^1_L (or more generally H^p_L), see e.g. [3–5,9].

The methods we use are based on [5], in which the authors proved a Hörmander-type multiplier theorem on H^1_L. The result for semigroups (Corollary 3.2) is a consequence of Theorem 3.1, which treats dilations of more general multipliers than $e^{-\lambda}$. Finally, using Theorem 3.1 we also prove the strong H^1_L continuity of the group of imaginary powers $\{L^{iu}\}_{u\in\mathbb{R}}$, see Corollary 3.3.

2. Preliminaries

Let $(\Omega, d(x, y))$ be a metric space equipped with a positive measure μ. We assume that (Ω, d, μ) is a space of homogeneous type in the sense of Coifman–Weiss [1], that is, there exists a constant $C > 0$ such that

$$\mu(B_d(x, 2t)) \le C\mu(B_d(x, t)) \quad \text{for every } x \in \Omega,\ t > 0, \tag{2.1}$$

where $B_d(x, t) = \{y \in \Omega : d(x, y) < t\}$. The condition (2.1) implies that there exist constants $C_0 > 0$ and $q > 0$ such that

$$\mu(B_d(x, st)) \le C_0 s^q \mu(B_d(x, t)) \quad \text{for every } x \in \Omega,\ t > 0,\ s > 1. \tag{2.2}$$

In what follows we set n_0 to be the infimum over q in (2.2).

Let $\{e^{-tL}\}_{t>0}$ be a semigroup of linear operators on $L^2(\Omega, d\mu)$ generated by $-L$. Here L is a non-negative, self-adjoint operator with domain $\mathcal{D}(L)$. For $M \in \mathbb{N}$, by $\mathcal{D}(L^M)$ we mean the domain of L^M given by the spectral theorem. We assume additionally that L is injective on $\mathcal{D}(L)$. Throughout the paper we impose that $T_t := e^{-tL}$ satisfies Davies–Gaffney estimates, that is,

$$|\langle T_t f_1, f_2\rangle| \le C \exp\left(-\frac{\text{dist}\,(U_1, U_2)^2}{ct}\right) \|f_1\|_{L^2(\Omega)}\|f_2\|_{L^2(\Omega)} \tag{2.3}$$

for every $f_i \in L^2(\Omega)$, supp $f_i \subset U_i$, $i = 1, 2$, U_i are open subsets of Ω.