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The approximation of almost time- and band-limited functions by their expansion in some orthogonal polynomials bases

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Abstract

The aim of this paper is to investigate the quality of approximation of almost time- and almost band-limited functions by its expansion in two classical orthogonal polynomials bases: the Hermite basis and the ultraspherical polynomials bases (which include Legendre and Chebyshev bases as particular cases). As a corollary, this allows us to obtain the quality of approximation in the L^2-Sobolev space by these orthogonal polynomials bases. Also, we obtain the rate of convergence of the Legendre series expansion of the prolate spheroidal wave functions.

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Keywords: Almost time and band limited functions; Hermite functions; Legendre polynomials; Chebyshev polynomials; Ultraspherical polynomials; Prolate spheroidal wave functions

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1. Introduction

The aim of this paper is to obtain the speed of convergence of the Hermite and the ultraspherical expansion of L^2 functions that are almost time- and band-limited to fixed intervals.

Time-limited functions and band-limited functions play a fundamental role in signal and image processing. The time-limiting assumption is natural as a signal can only be measured over a finite duration. The band-limiting assumption is natural as well due to channel capacity limitations. It is also essential to apply sampling theory. Unfortunately, the simplest form of the uncertainty principle tells us that a signal cannot be simultaneously time- and band-limited. A natural assumption is thus that a signal is almost time- and almost band-limited in the following sense:

Definition. Let $T, \Omega > 0$ and $\varepsilon_T, \varepsilon_\Omega > 0$. A function $f \in L^2(\mathbb{R})$ is said to be

- ε_T-*almost time-limited* to $[-T, T]$ if

$$\int_{|t|>T} |f(t)|^2 \,\mathrm{d}t \le \varepsilon_T^2 \, \|f\|_{L^2(\mathbb{R})}^2 ;$$

- ε_Ω-*almost band-limited* to $[-\Omega, \Omega]$ if

$$\int_{|\omega|>\Omega} |\widehat{f}(\omega)|^2 \,\mathrm{d}\omega \le \varepsilon_\Omega^2 \, \|f\|_{L^2(\mathbb{R})}^2 .$$

Here and throughout this paper the Fourier transform is normalized so that, for $f \in L^1(\mathbb{R})$,

$$\widehat{f}(\omega) := \mathcal{F}[f](\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\omega} \,\mathrm{d}t.$$

Of course, given $f \in L^2(\mathbb{R})$, for every $\varepsilon_T, \varepsilon_\Omega > 0$ there exist $T, \Omega > 0$ such that f is ε_T-almost time-limited to $[-T, T]$ and ε_Ω-almost band-limited to $[-\Omega, \Omega]$. The point here is that we consider $T, \Omega, \varepsilon_T, \varepsilon_\Omega$ as fixed parameters. A typical example we have in mind is that $f \in H^s(\mathbb{R})$ and is time-limited to $[-T, T]$. Such an hypothesis is common in tomography, *see e.g.* [21]. Now, if $f \in H^s(\mathbb{R})$ with $s > 0$, that is, if

$$\|f\|_{H^s(\mathbb{R})}^2 := \int_{\mathbb{R}} (1 + |\omega|)^{2s} |\widehat{f}(\omega)|^2 \,\mathrm{d}\omega < +\infty,$$

then

$$\int_{|\omega|>\Omega} |\widehat{f}(\omega)|^2 \,\mathrm{d}\omega \le \int_{|\omega|>\Omega} \frac{(1+|\omega|)^{2s}}{(1+|\Omega|)^{2s}} |\widehat{f}(\omega)|^2 \,\mathrm{d}\omega \le \frac{\|f\|_{H^s(\mathbb{R})}^2}{(1+|\Omega|)^{2s}}.$$

Thus f is $\frac{1}{(1+|\Omega|)^s} \frac{\|f\|_{H^s}}{\|f\|_{L^2(\mathbb{R})}}$-almost band-limited to $[-\Omega, \Omega]$.

The investigation of the set of almost time- and band-limited functions has been initially carried through Landau, Pollak and Slepian in the 1960s and is now commonly called the Bell Lab Program in applied harmonic analysis. Thanks to their seminal work, the optimal orthogonal system for representing almost time- and band-limited functions is known. The system in question consists of the so called prolate spheroidal wave functions (PSWFs) ψ_k^T, and has many valuable properties (see [16,17,28,26,27]). Note that these PSWFs form the infinite and countable set of the eigenfunctions of a time–frequency limiting operator. More precisely, for two measurable sets $\mathbf{T}, \mathbf{\Omega}$, let $D(\mathbf{T}) = \{f \in L^2(\mathbb{R}) : \operatorname{supp} f \subset \mathbf{T}\}$ be the set of $\mathbf{T}$-time-limited