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Full length article

On the log-concavity of the fractional integral of the sine function

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Abstract

We prove that the function

$$F_\lambda(x) := \int_0^x (x-t)^\lambda \sin t \, dt$$

is logarithmically concave on $(0, \infty)$ if and only if $\lambda \geq 2$. As a consequence, a Turán type inequality for certain Lommel functions of the first kind is obtained. Furthermore, some monotonicity properties of functions involving the remainders of the Taylor series expansion of the functions $\sin x$ and $\cos x$ are given.

MSC: primary 33C10; 33B10; secondary 26D15; 33B15

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1. Introduction and results

A function $f : I \to (0, \infty)$ is called logarithmically concave (or log-concave, for short) on the interval I if $\log f$ is a concave function on I. If f is twice differentiable, the log-concavity

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of f on I is equivalent to $[f'(x)/f(x)]' \leq 0$ and, in turn, $f''(x)\,f(x) - [f'(x)]^2 \leq 0$ for all $x \in I$. Clearly, every positive and concave function is log-concave. The product of log-concave functions is log-concave, too. However, the sum of log-concave functions is not, in general, log-concave.

Log-concave functions appear frequently in many problems of classical analysis, probability theory and convex optimization. As it happens, many common probability distributions are log-concave [6]. The log-concavity of probability densities and of integrals involving probability densities has interesting qualitative implications in many areas of economics, in political science, in biology and in industrial engineering [4]. For further background information and applications of log-concave functions in both discrete and continuous setting, we refer to the recent survey paper [13].

Let $f : [0, \infty) \to \mathbb{R}$ be a locally integrable function. The fractional integral I_α, $\alpha > 0$, of f is defined by the formula

$$(I_\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, f(t)\, dt,$$

where $\Gamma(\alpha)$ is Euler's Gamma function. We refer to [3, p. 111] and [11, p. 98] for the definition, properties and applications of fractional integrals in the theory of special functions.

For $\lambda > 0$ we consider the fractional integral

$$F_\lambda(x) := \int_0^x (x-t)^\lambda\, \sin t\, dt, \quad x > 0.$$

It should be mentioned that this is a positive function for all $x > 0$ and that $F_\lambda(x)$ can also be defined for $-1 < \lambda \leq 0$, but it is not strictly positive on $(0, \infty)$ for this range of λ, see Section 2.

The main result of this paper is the following.

Theorem 1.1. *The function $F_\lambda(x)$ is logarithmically concave on $(0, \infty)$, that is,*

$$F''_\lambda(x)\, F_\lambda(x) - [F'_\lambda(x)]^2 \leq 0, \quad \textit{for all } x > 0, \tag{1.1}$$

precisely when $\lambda \geq 2$. For $\lambda \geq 2$, equality occurs in (1.1) only when $\lambda = 2$ and $\tan \frac{x}{2} = \frac{x}{2}$.

We observe that

$$F_\lambda(x) = x^{\lambda+1} \int_0^1 (1-t)^\lambda\, \sin xt\, dt, \tag{1.2}$$

from which it follows that $F_\lambda(x)$ is infinitely often differentiable on $(0, \infty)$ for $\lambda > -1$.

We also have

$$F_\lambda(x) = \int_0^x t^\lambda\, \sin(x-t)\, dt = \sqrt{x}\, s_{\lambda+\frac{1}{2},\, \frac{1}{2}}(x), \tag{1.3}$$

where $s_{\mu,\nu}(z)$ is the Lommel function of the first kind. We recall that $s_{\mu,\nu}(z)$ is a particular solution of the inhomogeneous Bessel differential equation

$$z^2 y'' + zy' + (z^2 - \nu^2)y = z^{\mu+1}.$$

It can be expressed in terms of a hypergeometric series

$$s_{\mu,\nu}(z) = \frac{z^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\, {}_1F_2\Big(1; \frac{\mu-\nu+3}{2}, \frac{\mu+\nu+3}{2}; -\frac{z^2}{4}\Big).$$