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Journal of Approximation Theory 210 (2016) 41–79

Journal of Approximation Theory

www.elsevier.com/locate/jat

Full length article

Asymptotic behaviour of some families of orthonormal polynomials and an associated Hilbert space

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Received 23 February 2015; received in revised form 19 May 2016; accepted 16 June 2016
Available online 2 July 2016

Communicated by Paul Nevai

Abstract

We characterise asymptotic behaviour of families of symmetric orthonormal polynomials whose recursion coefficients satisfy certain conditions, satisfied for example by the (normalised) Hermite polynomials. More generally, these conditions are satisfied by the recursion coefficients of the form $c(n+1)^p$ for $0 < p < 1$ and $c > 0$, as well as by recursion coefficients which correspond to polynomials orthonormal with respect to the exponential weight $W(x) = \exp(-|x|^\beta)$ for $\beta > 1$. We use these results to show that, in a Hilbert space defined in a natural way by such a family of orthonormal polynomials, every two complex exponentials $e_\omega(t) = \mathrm{e}^{i\omega t}$ and $e_\sigma(t) = \mathrm{e}^{i\sigma t}$ of distinct frequencies ω, σ are mutually orthogonal. We finally formulate a surprising conjecture for the corresponding families of non-symmetric orthonormal polynomials; extensive numerical tests indicate that such a conjecture appears to be true.

MSC: 42C05; 41A60; 42A75

Keywords: Orthogonal polynomials; Unbounded recurrence coefficients; Christoffel functions; Almost periodic functions; Signal processing

1. Introduction

Let $\gamma_n > 0$ for $n \geq 0$ be the recursion coefficients that correspond to a symmetric positive definite family of orthonormal polynomials $(p_n : n \in \mathbb{N})$. Thus, $p_0(\omega) = 1$ and if we set $\gamma_{-1} = 1$

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http://dx.doi.org/10.1016/j.jat.2016.06.002
0021-9045/

and $p_{-1}(\omega) = 0$, then the three term recurrence

$$\gamma_n \, p_{n+1}(\omega) = \omega \, p_n(\omega) - \gamma_{n-1} \, p_{n-1}(\omega) \tag{1}$$

holds for all $n \geq 0$.[1] Let also s_n be the first and d_n the second order forward finite differences of these recursion coefficients:

$$s_n = \gamma_{n+1} - \gamma_n; \qquad d_n = s_{n+1} - s_n.$$

We consider families of orthonormal polynomials such that the corresponding recursion coefficients γ_n satisfy the following conditions.

($\mathcal{C}_1$) $\gamma_n \to \infty$; ($\mathcal{C}_2$) $s_n \to 0$;
($\mathcal{C}_3$) There exist n_0, m_0 such that $\gamma_{n+m} > \gamma_n$ holds for all $n \geq n_0$ and all $m \geq m_0$.

A sequence γ_n which satisfies condition ($\mathcal{C}_3$) will be called an *almost increasing sequence*; an *almost decreasing sequence* is defined in an analogous way. Clearly, every increasing sequence is also an almost increasing sequence with $n_0 = 0$ and $m_0 = 1$.

($\mathcal{C}_4$) $\displaystyle\sum_{j=0}^{\infty} \frac{1}{\gamma_j} = \infty$; ($\mathcal{C}_5$) there exists $\kappa > 1$ such that $\displaystyle\sum_{j=0}^{\infty} \frac{1}{\gamma_j^{\kappa}} < \infty$;
($\mathcal{C}_6$) $\displaystyle\sum_{n=0}^{\infty} \frac{|s_n|}{\gamma_n^2} < \infty$; ($\mathcal{C}_7$) $\displaystyle\sum_{n=0}^{\infty} \frac{|d_n|}{\gamma_n} < \infty$.

Note that if the Hermite polynomials are normalised into a corresponding orthonormal family with respect to the weight $W(x) = \mathrm{e}^{-x^2}/\sqrt{\pi}$, then their recursion coefficients are of the form $\gamma_n = (n+1)^{1/2}/\sqrt{2}$.

Lemma 1. *Conditions* ($\mathcal{C}_1$)–($\mathcal{C}_7$) *are satisfied by the Hermite polynomials, and more generally,*

(a) *by families with recursion coefficients of the form* $\gamma_n = c(n+1)^p$ *for any* $0 < p < 1$ *and* $c > 0$;
(b) *by families orthonormal with respect to the exponential weight* $W(\omega) = \exp(-c|\omega|^{\beta})$ *for* $\beta > 1$ *and* $c > 0$.

Proof. (a) If $p > 0$ then γ_n are increasing and $\gamma_n \to \infty$; moreover, since for $\gamma_n = c(n+1)^p$ all forward finite differences $\Delta^k(n)$ satisfy $\Delta^k(n) = \mathcal{O}\left(n^{p-k}\right)$, we obtain

$$\sum_{n=0}^{\infty} \frac{s_n}{\gamma_n^2} = \sum_{n=0}^{\infty} \mathcal{O}\left(\frac{n^{p-1}}{n^{2p}}\right) = \mathcal{O}\left(\sum_{n=0}^{\infty} n^{-p-1}\right) < \infty.$$

On the other hand, if $p < 1$ then $s_n = \mathcal{O}\left(n^{p-1}\right) \to 0$ and ($\mathcal{C}_4$) holds; also,

$$\sum_{n=0}^{\infty} d_n = \mathcal{O}\left(\sum_{n=0}^{\infty} n^{p-2}\right) < \infty. \tag{2}$$

Note that (2) is stronger than what is required by condition ($\mathcal{C}_7$). Finally, if $0 < p < 1$ then ($\mathcal{C}_5$) holds for every $\kappa > 1/p$.

1 See, for example, [2].