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Positivity and Fourier integrals over regular hexagon

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Abstract

Let $f \in L^1(\mathbb{R}^2)$ and let $\widehat{f}$ be its Fourier integral. We study summability of the partial integral $S_{\rho,\mathsf{H}}(x) = \int_{\{\|y\|_{\mathsf{H}} \le \rho\}} e^{ix\cdot y} \widehat{f}(y)dy$, where $\|y\|_{\mathsf{H}}$ denotes the uniform norm taken over the regular hexagonal domain. We prove that the Riesz (R, δ) means of the inverse Fourier integrals are nonnegative if and only if $\delta \ge 2$. Moreover, we describe a class of $\|\cdot\|_{\mathsf{H}}$-radial functions that are positive definite on $\mathbb{R}^2$.

MSC: 42B08; 41A25; 41A63

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1. Introduction

The classical Bochner–Riesz means of the Fourier integral have kernels that are radial functions, or the $\|\cdot\|_2$-radial functions, where $\|\cdot\|_2$ denotes the usual Euclidean norm. We study their analogues that have kernels being $\|\cdot\|_{\mathsf{H}}$-radial functions, where $\|\cdot\|_{\mathsf{H}}$ denotes the uniform norm of the regular hexagonal domain of $\mathbb{R}^2$, and $\|\cdot\|_{\mathsf{H}}$-radial functions that are positive definite functions on $\mathbb{R}^2$.

Let f be a function in $L^1(\mathbb{R}^d)$. The Fourier transform $\widehat{f}$ and its inverse are defined by

$$\widehat{f}(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix\cdot y} f(x)dx \quad \text{and} \quad f(x) = \int_{\mathbb{R}^d} e^{iy\cdot x} \widehat{f}(y)dy, \tag{1.1}$$

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where the latter integral need not exist for an arbitrary function $f \in L^1(\mathbb{R}^d)$, a fact that motivates the study of summability methods. The classical Bochner–Riesz means (cf. [6]) of the inverse Fourier transform are defined by

$$S^{(2)}_{R,\delta} f(x) = \int_{\|y\|_2 \le R} \left(1 - \frac{\|y\|_2^2}{R^2}\right)^{\delta} e^{iy\cdot x} \widehat{f}(y) dy. \tag{1.2}$$

The convergence of these means has been studied extensively. If $\|y\|_2$ is replaced by the ℓ_1 norm $|y|_1 := |y_1| + \cdots + |y_d|$ in (1.2), we denote the new means by $S^{(1)}_{R,\delta} f$ and call them ℓ_1-Riesz (R, δ) means. It was proved in [2] that, in ℓ_1 summability, the (R, δ) means $S^{(1)}_{R,\delta} f$ define positive linear transformations on $L^1(\mathbb{R}^d)$ exactly when $\delta \ge 2d - 1$. In contrast, in ℓ_2 summability, the Bochner–Riesz means do not define positive transformations for any $\delta > 0$ [4].

In the present paper we study the case when $\|\cdot\|_2$ in (1.2) is replaced by the uniform norm $\|\cdot\|_{\mathsf{H}}$ over the regular hexagonal domain in $\mathbb{R}^2$. In this case it is more convenient to work in homogeneous coordinates of

$$\mathbb{R}^3_{\mathsf{H}} := \{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3 : t_1 + t_2 + t_3 = 0\},$$

for which the regular hexagonal domain is equivalent to $\{\mathbf{t} \in \mathbb{R}^3_{\mathsf{H}} : \|\mathbf{t}\|_{\mathsf{H}} \le 1\}$, where

$$\|\mathbf{t}\|_{\mathsf{H}} := \max_{1 \le i \le 3} |t_i|.$$

In $\mathbb{R}^3_{\mathsf{H}}$ the Fourier transform and its inverse can be defined by

$$\widehat{f}(\mathbf{s}) = \frac{1}{3\pi^2} \int_{\mathbb{R}^3_{\mathsf{H}}} e^{-\frac{2i}{3}\mathbf{t}\cdot\mathbf{s}} f(\mathbf{t}) d\mathbf{t} \quad \text{and} \quad f(\mathbf{t}) = \int_{\mathbb{R}^3_{\mathsf{H}}} e^{\frac{2i}{3}\mathbf{t}\cdot\mathbf{s}} \widehat{f}(\mathbf{s}) d\mathbf{s}, \tag{1.3}$$

as we shall see in the next section. The Riesz (R, δ) means then become

$$S_{R,\delta} f(\mathbf{t}) := \int_{\|\mathbf{s}\|_{\mathsf{H}} \le R} \left(1 - \frac{\|\mathbf{s}\|_{\mathsf{H}}}{R}\right)^{\delta} e^{i\mathbf{s}\cdot\mathbf{t}} \widehat{f}(\mathbf{s}) d\mathbf{s}. \tag{1.4}$$

The symmetry of the regular hexagonal domain makes it possible to derive a close form for the Dirichlet kernel,

$$D_R(\mathbf{t}) = \int_{\|\mathbf{t}\|_{\mathsf{H}} \le R} e^{\frac{2i}{3}\mathbf{s}\cdot\mathbf{t}} d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^3_{\mathsf{H}},$$

which can be used to establish the following theorem.

Theorem 1.1. *For $\delta \ge 2$ the Riesz (R, δ) means of the hexagonal partial integral (1.4) define positive linear transformations on $L^1(\mathbb{R}^3_{\mathsf{H}})$; the order of the summability to assure positivity is best possible.*

We note that the minimal order of the summability to assure positivity of the Riesz (R, δ) means for ℓ_1 summability is $\delta \ge 3$ when $d = 2$.

A function $\phi : \mathbb{R}^3_{\mathsf{H}} \mapsto \mathbb{R}$ is called $\|\cdot\|_{\mathsf{H}}$ invariant, or $\|\cdot\|_{\mathsf{H}}$-radial, if $\phi(\mathbf{t}) = \phi_0(\|\mathbf{t}\|_{\mathsf{H}})$ for some $\phi_0 : \mathbb{R}_+ := [0, \infty) \mapsto \mathbb{R}$. Although the Dirichlet kernel is not $\|\cdot\|_{\mathsf{H}}$ radial, it has additional structure that allows us to characterize positive definiteness of such functions.