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Chebyshev sets in geodesic spaces

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Abstract

In this paper we study several properties of Chebyshev sets in geodesic spaces. We focus on analyzing if some well-known results that characterize convexity of such sets in Hilbert spaces are also valid in the setting of geodesic spaces with bounded curvature.

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1. Introduction

Let (X, d) be a metric space and $C \subseteq X$. The *metric projection* P_C onto C is the mapping $P_C : X \to 2^C$ defined by

$$P_C(z) = \{y \in C : d(z, y) = \text{dist}(z, C)\} \quad \text{for every } z \in X,$$

where $\text{dist}(z, C) = \inf_{y \in C} d(z, y)$. If $P_C(x)$ is a singleton for every $x \in X$, then the set C is called *Chebyshev*. This concept was introduced by Efimov and Stechkin [15] and stems from a

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famous result in approximation theory which goes back to Chebyshev [12] and asserts that in the space $C[0, 1]$, the subspace of polynomials of degree no larger than n is a Chebyshev set. The term metric projection was coined by Aronszajn and Smith in [3] although it already implicitly appeared in a result proved earlier by Riesz [31] which states that every closed convex subset of a Hilbert space is Chebyshev. The converse was first proved by Bunt [9] in finite-dimensional Hilbert spaces. Shortly afterwards, this result was extended to some classes of finite-dimensional normed linear spaces [11,23,35,34]. In [21], Johnson provided an example of a nonconvex Chebyshev set in an incomplete infinite-dimensional inner product space. So far, attempts to generalize this construction to a Hilbert space have been unsuccessful and the question concerning the convexity of Chebyshev sets in Hilbert spaces is still open.

The search for additional conditions, which, imposed on Chebyshev sets, ensure their convexity, was one of the most fruitful research directions in approximation theory during the second half of the last century. It has been proved by different authors that concepts such as approximative compactness, continuity of the metric projection or sun (see Section 2 for other related properties and precise definitions) play a key role in determining the convexity of a Chebyshev set in the setting of linear spaces. We include below a result (see [13, Theorem 2]) which summarizes some of the main progress in this direction in the context of Hilbert spaces. Historical notes and subtle hints about the different equivalences gathered in the next theorem can be found in the monographs by Singer [33], Braess [7] or Deutsch [14].

Theorem 1.1. *Let H be a Hilbert space and $C \subseteq X$ a Chebyshev set. Each of the following statements is equivalent to C being convex:*

(i) *C is weakly closed (Klee* [23]*);*
(ii) *C is approximatively compact (Efimov–Stechkin* [16]*);*
(iii) *P_C is continuous (Vlasov* [36] *and Asplund* [4]*);*
(iv) *P_C is radially continuous (Vlasov* [37]*);*
(v) *for every $x \in H \setminus C$,*

$$\lim_{\varepsilon\to 0^+} \frac{\operatorname{dist}(x_\varepsilon, C) - \operatorname{dist}(x, C)}{\|x_\varepsilon - x\|} = 1,$$

where $x_\varepsilon = x + \varepsilon[x - P_C(x)]$ (Vlasov [37]*);*
(vi) *C is a sun (Efimov–Stechkin* [16]*);*
(vii) *P_C is nonexpansive (Phelps* [29]*).*

Another property that has been studied in connection to convexity of Chebyshev sets is bounded compactness. In Banach spaces, any boundedly compact Chebyshev set is a sun (see [35], also [38, Theorem 4.13] for several conditions which ensure that in a Banach space a Chebyshev set is a sun). Thus, by [38, Theorem 3.9], in smooth Banach spaces, any boundedly compact Chebyshev set is convex. However, bounded compactness and convexity of Chebyshev sets are not equivalent since there exist convex Chebyshev sets that are not boundedly compact such as the closed unit ball in ℓ_2.

Geodesic metric spaces constitute a natural generalization of manifolds. Alexandrov [1] introduced the notion of lower and upper curvature bounds for geodesic spaces in terms of comparison properties for geodesic triangles. These notions coincide with the corresponding ones for the sectional curvature when the metric space is a complete simply connected Riemannian manifold. It is well-known that bounding the sectional curvature is essential for many results in geometric analysis as in the theory of generalized harmonic maps developed by Jost in [22]. In this seminal