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Completeness of Gabor systems

Karlheinz Gröchenig*, Antti Haimi, José Luis Romero

Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Vienna, Austria

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Abstract

We investigate the completeness of Gabor systems with respect to several classes of window functions on rational lattices. Our main results show that the time–frequency shifts of every finite linear combination of Hermite functions with respect to a rational lattice are complete in $L^2(\mathbb{R})$, thus generalizing a remark of von Neumann (and proved by Bargmann, Perelomov et al.). An analogous result is proven for functions that factor into certain rational functions and the Gaussian. The results are also interesting from a conceptual point of view since they show a vast difference between the completeness and the frame property of a Gabor system. In the terminology of physics we prove new results about the completeness of coherent state subsystems.

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1. Introduction

We study the question when the set of time–frequency shifts

$$\mathcal{G}(g, \alpha, \beta) = \{e^{2\pi i \beta l x} g(x - \alpha k) : k, l \in \mathbb{Z}\} \tag{1}$$

* Corresponding author.
E-mail addresses: karlheinz.groechenig@univie.ac.at (K. Gröchenig), antti.haimi@univie.ac.at (A. Haimi), jose.luis.romero@univie.ac.at (J.L. Romero).

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is complete in $L^2(\mathbb{R})$, where $g \in L^2(\mathbb{R})$ and the lattice parameters $\alpha, \beta > 0$ are fixed. The completeness question arose first in J. von Neumann's treatment of quantum mechanics [29] and remains relevant in physics and in applied mathematics. The motivation in signal analysis and time–frequency analysis comes from Gabor's fundamental paper [14] on information theory. Gabor tried to expand a given function (signal) into a series of time–frequency shifts. Correspondingly, in mathematical terminology the set $\mathcal{G}(g, \alpha, \beta)$ is called a Gabor system with window function g. In quantum mechanics the functions $e^{2\pi i \beta l x} g(x - \alpha k)$ are called phase-space shifts of a (generalized) coherent state, and $\mathcal{G}(g, \alpha, \beta)$ can be interpreted as a discrete set of coherent states with respect to the Heisenberg group over a lattice $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ [3]. In fact, Perelomov's book [31] on coherent states contains several sections devoted to the "completeness of coherent state subsystems".

In the applied mathematics literature of the last 20 years the interest in Gabor systems has shifted to the frame property, mainly for numerical reasons [12] and because Gabor frames can be used to characterize function spaces [11] and to describe pseudodifferential operators [16]. Here $\mathcal{G}(g, \alpha, \beta)$ is a (Gabor) frame for $L^2(\mathbb{R})$, if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{k,l \in \mathbb{Z}} |\langle f, e^{2\pi i \beta l \cdot} g(\cdot - \alpha k)\rangle|^2 \leq B\|f\|_2^2 \quad \forall f \in L^2(\mathbb{R}). \tag{2}$$

Clearly (2) implies that $\mathcal{G}(g, \alpha, \beta)$ is complete in $L^2(\mathbb{R})$, but in general the frame property of $\mathcal{G}(g, \alpha, \beta)$ is much stronger than its completeness. This difference is already present in von Neumann's example $\mathcal{G}(\phi, 1, 1)$ where $\phi(x) = e^{-\pi x^2}$ is the Gaussian, i.e., the canonical coherent state in quantum mechanics, and $\alpha = \beta = 1$. In this case it was proved 40 years after von Neumann that $\mathcal{G}(\phi, 1, 1)$ is complete, but not a frame [6,30].

The main intuition for the results about the completeness and the frame property of $\mathcal{G}(g, \alpha, \beta)$ is based on the uncertainty principle. According to the uncertainty principle every physical state g, i.e., $g \in L^2(\mathbb{R})$, $\|g\|_2^2 = \int_{\mathbb{R}} |g(x)|^2\, dx = 1$, occupies a cell in phase space (in the time–frequency plane) of minimal area one. The phase-space shift $e^{2\pi i \beta l x} g(x - \alpha k)$ is located roughly at position αk and momentum βl in phase space $\mathbb{R}^2$. Thus, in order to cover the entire phase space with a discrete set of coherent states $\mathcal{G}(g, \alpha, \beta)$, we must have necessarily $\alpha\beta \leq 1$, otherwise there would be gaps in phase space that cannot be reached by a phase-space shift of the form $e^{2\pi i \beta l x} g(x - \alpha k)$. The physical intuition has been made mathematically rigorous in the form of numerous density theorems for Gabor systems: *If $\mathcal{G}(g, \alpha, \beta)$ is complete in $L^2(\mathbb{R})$, then necessarily $\alpha\beta \leq 1$* [9,32,22]. The converse holds only for special window functions. It was already noted in [6,30] that for the Gaussian $\phi(x) = e^{-\pi x^2}$ the set $\mathcal{G}(\phi, \alpha, \beta) = \{e^{2\pi i \beta l x} e^{-(x-k\alpha)^2} : k, l \in \mathbb{Z}\}$ is complete for $\alpha\beta \leq 1$ and incomplete for $\alpha\beta > 1$. In 1992 Lyubarskii [26] and Seip [34] strengthened this statement and showed that $\mathcal{G}(\phi, \alpha, \beta)$ is frame, if and only if $\alpha\beta < 1$. (In fact, they stated their results for arbitrary time–frequency shifts, not just lattice shifts.) The recent work [20] shows that an analogous result also holds for the class of so-called totally positive functions of finite type. Again, as in the case of the Gaussian, these functions possess good time–frequency localization, which in mathematical terms amounts to additional analyticity properties.

In this paper we return to the completeness problem for Gabor systems. In agreement with the physical description of quantum states, we will assume that the window functions possess strong localization properties in the time–frequency plane. Technically, we will assume that g and its Fourier transform have exponential decay.