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Measures for orthogonal polynomials with unbounded recurrence coefficients

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Abstract

Systems of orthogonal polynomials whose recurrence coefficients tend to infinity are considered. A summability condition is imposed on the coefficients and the consequences for the measure of orthogonality are discussed. Also discussed are asymptotics for the polynomials.

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1. Introduction

Let $\{p_n(x)\}_{n=-1}^{\infty}$ be a system of polynomials satisfying the recurrence relations

$$a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x) = xp_n(x), \qquad p_0(x) = 1, \qquad p_{-1}(x) = 0, \quad (1)$$

with $a_{n+1} > 0$ and b_n real for $n = 0, 1, \ldots$. By the spectral theorem for orthogonal polynomials these polynomials are orthonormal with respect to some positive probability measure supported

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on the real line. There has been much work done on the asymptotics or spectral properties of polynomials whose recurrence coefficients are unbounded [7,8,14,22] and here we are interested in the problem of constructing the orthogonality measure given the coefficients in the recurrence formula. This problem for the bounded case has been an area of ongoing intense investigation due to its connection to the discrete Schrödinger equation [4,6,10–13,16,19–21,23,25,26] and to the connection between the continuum limits of the recurrence relations with varying recurrence coefficients and discrete integrable systems [5,18]. However with regard to the construction of the measure of orthogonality almost all the results are for cases with bounded recurrence coefficients. Unfortunately many of the techniques developed for the bounded case cannot be applied to the unbounded case. Important results in this direction were obtained by Chihara [2,3]. These results show how delicate the semi-bounded case it with regard to the location of the spectrum of the Jacobi matrix. Careful analysis of the work of Máté–Nevai [20], Máté–Nevai–Totik [21], and Van Assche–Geronimo [26] for coefficients that are of bounded variation i.e.

$$\sum_{n=1}^{\infty} |a_{n+1} - a_n| + |b_n - b_{n-1}| < \infty$$

with limits $a_n \to \frac{1}{2}$ and $b_n \to 0$ reveals that it is possible to modify these techniques so that they apply to certain cases when the recurrence coefficients tend to infinity. For bounded recurrence coefficients obeying the above criteria the absolutely continuous part of the orthogonality measure is given by

$$du_{ac} = \frac{2}{\pi} \frac{\sqrt{1-x^2}dx}{|\xi(x)|^2 \prod_{n=1}^{\infty} |\zeta_n(x)|^2}, \tag{2}$$

where

$$\zeta_n(x) := \frac{x - b_n + \sqrt{(x-b_n)^2 - 4a_n^2}}{2a_n},$$

is the mapping function $z = \zeta_n(x)$ of $\overline{\mathbb{C}} \setminus (\alpha_n,\ \beta_n)$ on $\overline{\mathbb{C}} \setminus \{|z| \le 1\}$ normalized as $\zeta_n(x)/x > 0$ when $x \to \infty$, and for $\xi(x)$ we have the expressions

$$\xi(x) = 1 + \sum_{k=1}^{\infty} \left\{ \frac{1}{\zeta_k(x)} - \frac{a_k/a_{k+1}}{\zeta_{k+1}(x)} \right\} \frac{p_{k-1}(x)}{\prod_{j=1}^{k} \zeta_j(x)},$$

or

$$\xi(x) = \lim_{n\to\infty} \frac{p_n(x) - \frac{b_n}{b_{n+1}} p_{n-1}(x)/\zeta_{n+1}(x)}{\prod_{j=1}^{n} \zeta_j(x)}.$$

An extension of this formula was used in [1] to show the connection between the limit of varying recurrence coefficients and their corresponding orthogonality measure and an analog of this formula for the unbounded case is necessary in order to extend the above results to the unbounded case. This is done in the next section.